Bounded context switching for valence systems

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Result and structure

Theorem

Reachability under bounded context switching

for valence systems over graph monoids

is always in NP *(for all graph monoids).*

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2. What are valence systems over graph monoids?
3. What is BCS for valence systems?

1. BCS

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Concurrent system, each component modeled as automaton

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Solution:
Consider bounded context switching (BCS)

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Useful as bugs *usually* occur within few context switches [MQ07,LPSZ08]

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Concurrent system where each component is a PDS, communicating via finite control

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Reachability under BCS is NP-complete

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Similar results for

- various types of components,
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For example:

Queues as storages [LMP08]

Pushdowns with dynamic thread creation [ABQ09]

Pushdowns communicating via queues [HLMS12]

...

Related work II

Our goal: General BCS result

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Our technique provides an algebraic view

2. Valence systems

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Introducing valence systems over some monoid $\mathbb{M}$

Monoid $\mathbb{M}$ represents the storage of the system

Syntax:

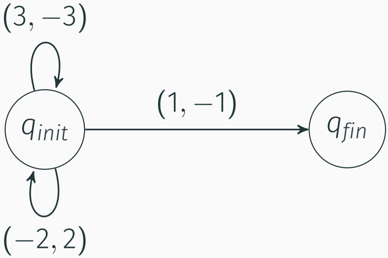
- Finite control
- Transitions labeled by generators of $\mathbb{M}$

Semantics:

- Configurations (q, m) with q control state, $m \in \mathbb{M}$
- Transition $q \xrightarrow{m'} q'$ leads to $(q', m \cdot m')$

Example

Valence system over $\mathbb{Z} \times \mathbb{Z}$ (with component-wise addition)



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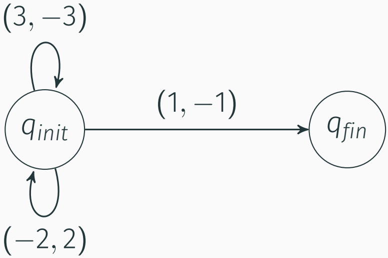

(essentially an integer 2-VASS)

Want results of the following shape:

Theorem

If monoid $\mathbb{M}$ satisfies condition c, then checking property P for all valence systems over $\mathbb{M}$ is in complexity class $\mathcal{C}$.

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For example, want classification of $P =$ reachability

Reachability for valence systems

Given: Valence system $\mathcal{A}$ over monoid $\mathbb{M}$

Decide: $(q_{init}, 1_{\mathbb{M}}) \to^* (q_{final}, 1_{\mathbb{M}})$?

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Monoid elements: Sequences of operations

modulo the congruence $o^+.o^- \cong \varepsilon$

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Valence systems over $\mathbb{M}_G$ are PDS over stack alphabet $\{a, b\}$

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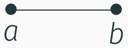
If $o \mathcal{I} u$, then $o^\pm$ and $u^\pm$ commute: $o^\pm.u^\pm \cong u^\pm.o^\pm$

Example: VASS



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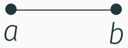

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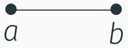

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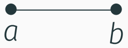

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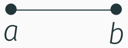

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Valence systems over $\mathbb{M}_G$ are 2-VASS

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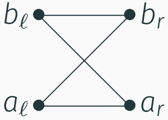

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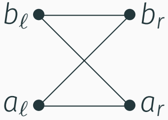

Any $m \in \{{a_\ell}^+, {a_\ell}^-, \ldots\}^*$ can be written as

$$m \cong m_{\upharpoonright_\ell} . m_{\upharpoonright_r}$$

such that $m \cong \varepsilon$ iff $m_{\upharpoonright_\ell} \cong \varepsilon$ and $m_{\upharpoonright_r} \cong \varepsilon$

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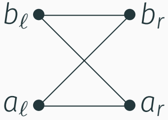

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Valence systems over $\mathbb{M}_G$ are 2-PDS (with a binary stack alphabet for each stack)

Graph monoids can model:

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Graph monoids cannot model:

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- Higher-order stacks

Results

Characterization results for valence systems/automata:

- reachability [Z15]
- regularity [Z11]
- context-freeness [BZ13]
- semilinearity of the Parikh image [BZ13]
- ...

3. BCS for valence systems

BCS for valence systems

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Concurrent system as valence system

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- The components share a control state
  (communication between components)

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We do not store the monoid element, but its syntactic representation

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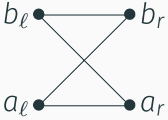
A slight modification

Consider configurations of the shape (q, m) where m is a sequence of operations

We do not store the monoid element, but its syntactic representation

Crucial as our notion of context is not invariant under congruence

Contexts



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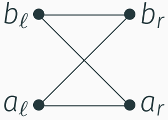

Nodes belonging to independent parts of the storage are connected by an edge

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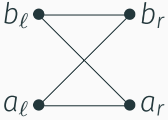

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$$m \;=\; \ldots o^{\pm}.u^{\pm} \ldots$$

with $o \, \mathcal{I} \, u$, then this constitutes a context switch

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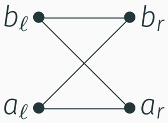

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In general, we need a more restrictive definition

Dependent computations

Definition

A sequence of operations m is called dependent

if for all $o^{\pm}, u^{\pm}$ in m with $o \neq u$, $o \mathcal{I} u$ **does not hold**.

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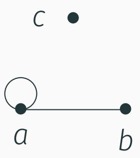

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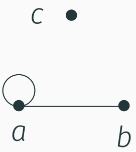

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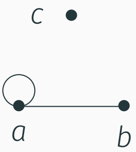

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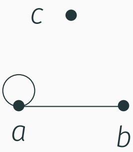

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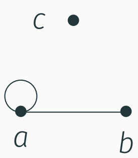

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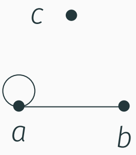

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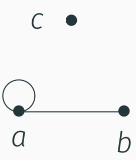

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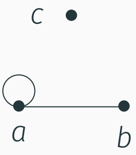

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	but $a^+a^- \cong a^+b^+b^-a^-$!

Contexts & context switches

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The i^{th} context of m is the maximal dependent prefix of m with the first $i-1$ contexts removed

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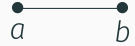
The number of context switches $cs(m)$ is the number of contexts minus 1

In the examples

Assume the number of context switches is bounded by k

PDS — no restriction



VASS — changing the counter $\leqslant k$ times

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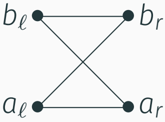

MPDS — changing the stack $\leqslant k$ times

The result

BCSREACH for valence systems over graph monoids

Given: Valence system $\mathcal{A}$ over $\mathbb{M}_G$, number k (in unary)

Decide: Is there $(q_{init}, \varepsilon) \to (q_{final}, m)$
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Theorem

BCSREACH for valence systems over graph monoids is in NP (for all graph monoids).

The proof / The algorithm

Proof outline

Need to find a computation $(q_{init}, \varepsilon) \to^* (q_{final}, m)$ with $m \cong \varepsilon$

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Then check existence of reducible block decomposition using guessing and representing blocks as finite automata

Block decomposition

If $m \cong \varepsilon$, then there is a reduction of m that

swaps letters

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E.g. $m_1.m_2 \to m_2.m_1$ if every symbol in m_1 commutes with every symbol in m_2

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> If $m \cong \varepsilon$, then its decomposition into letters is always freely reducible.
>
> Coarser decompositions might not be freely reducible:
>
> $$o^+u^+ \;,\; u^- \;,\; o^-$$

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Sequence is irreducible if it is not congruent to a shorter one

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Theorem

Let m be a sequence of operations with

- *k contexts*
- *each of them irreducible, and*
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Existence can be checked algorithmically

The algorithm

Given: valence system $\mathcal{A}$, bound k

The algorithm, Step I

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Part I: Enforcing irreducibility

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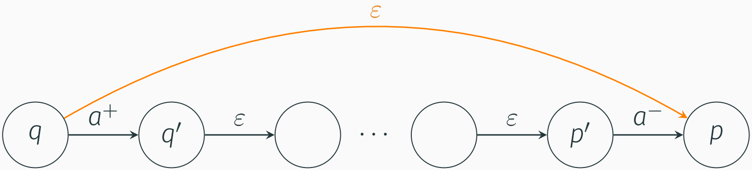
Part I: Enforcing irreducibility

1. Guess $\leqslant k$ dependent parts of $\mathcal{A}$
2. Saturate each part:



Let $\mathcal{A}_{\mathsf{sat}}$ be the resulting valence system

The algorithm, Step I

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Theorem

$(q_{init}, \varepsilon) \rightarrow^* (q_{final}, m)$ in $\mathcal{A}$ with $m \cong \varepsilon$ and $cs(m) \leqslant k$,

The algorithm, Step I

Let $\mathcal{A}_{\mathsf{sat}}$ be the resulting valence system

Theorem

$(q_{init}, \varepsilon) \to^* (q_{final}, m)$ *in* $\mathcal{A}$ *with* $m \cong \varepsilon$ *and* $cs(m) \leqslant k$,

iff

$(q_{init}, \varepsilon) \to^* (q_{final}, m')$ *in* $\mathcal{A}_{sat}$ *with* $m' \cong \varepsilon$, $cs(m') \leqslant k$,
and contexts of m' *irreducible.*

The algorithm, Step II

Part II:
Checking the existence of a freely reducible block decomposition

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3. For each context i, guess part of $\mathcal{A}_{\mathrm{sat}}$ that is used in block $m_{i,j}$ as NFA $\mathcal{A}_{i,j}$

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   Swap rule applicable to $\mathcal{A}_{i,j}$, $\mathcal{A}_{i',j'}$
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Cancel rule applicable to $\mathcal{A}_{i,j}$, $\mathcal{A}_{i',j'}$
if $\mathcal{L}(\mathcal{A}_{i,j}) \cap \mathcal{L}(\mathcal{A}_{i',j'})^{\text{inverse}}$ is non-empty

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If G^- is a clique, then BCSREACH(G) *is* NL-*complete.*

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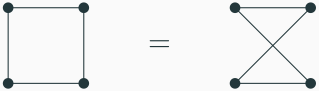
P4



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P4

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If G^- contains C4 as induced subgraph,
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Theorem

If G^- contains neither C4 nor P4 as induced subgraphs, then BCSREACH(G) *is in* P.

Conclusion

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+ almost complete classification of complexity for fixed graphs.

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Richer model supporting queues, higher order?

Thank you!

Questions?