Parity to Safety in Polynomial Time for Pushdown and Collapsible Pushdown Games

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Royal Holloway, University of London

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Abstract

- Safety games for pushdown systems:
  n-EXPTIME-complete
- Parity games for pushdown systems:
  n-EXPTIME-complete

(for order-n collapsible pushdown)

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We give a "natural" parity->safety reduction

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(for order-n collapsible pushdown)

We give a "natural" parity->safety reduction

(i.e. not parity algorithm -> TM -> safety)

Motivation

- Safety is easy to reason about (Parity is hard)
- Parity is more expressive
- To know the relationship

Ideas We Start With

- Finite-state parity to safety using counters (Bernet et al, 2002)

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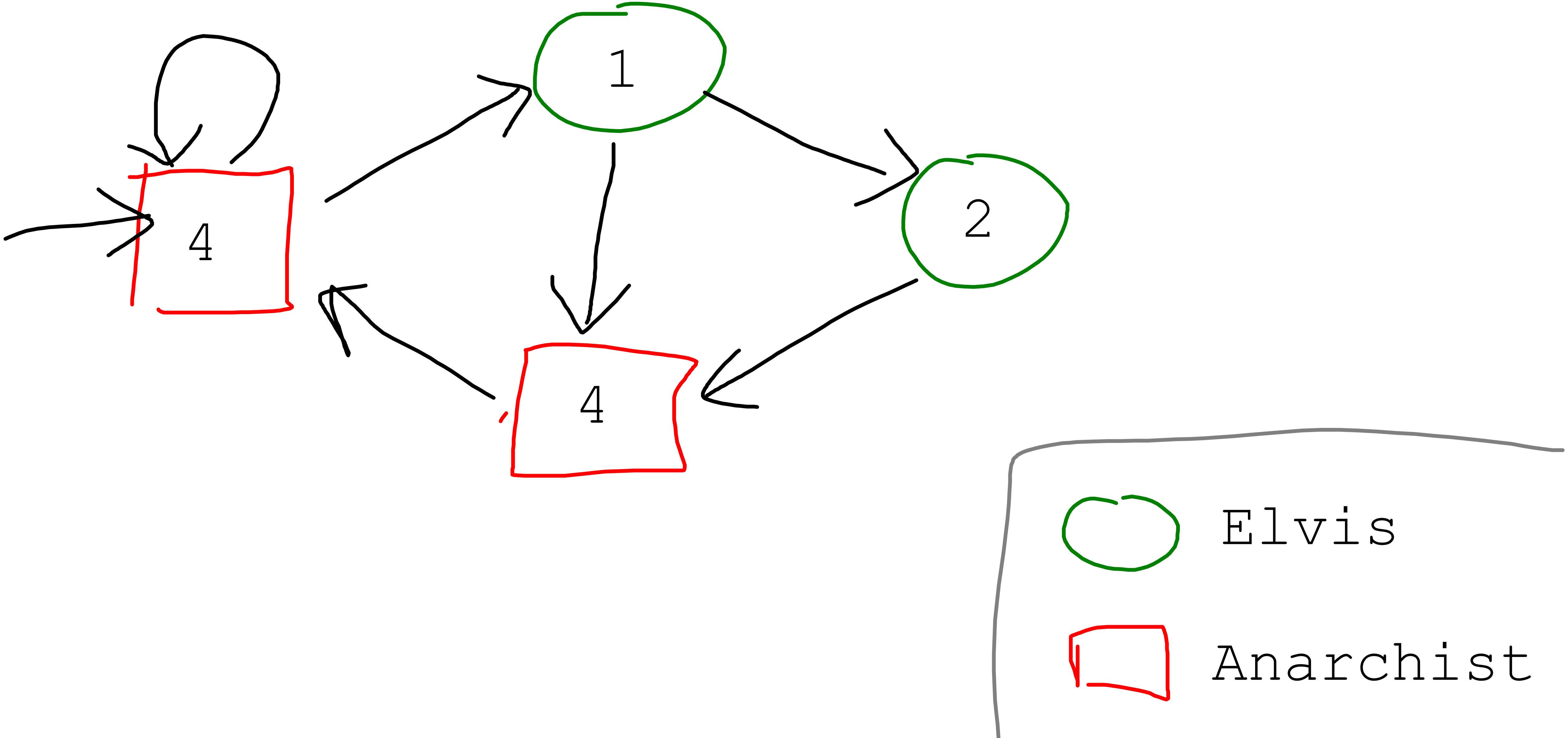
- Reduction order-n -> order-(n-1)
  - Rank awareness

  (H, Murawski, Ong, Serre, 2008)
- Encoding large counters in a pushdown stack (Cachat and Walukiewicz, 2007)

Contributions

- Generalise counter encoding to collapse
- Direct proof based on commutativity of
  - Counters encoding
  - Stack removal
- Counters behave like a stack

Parity Game



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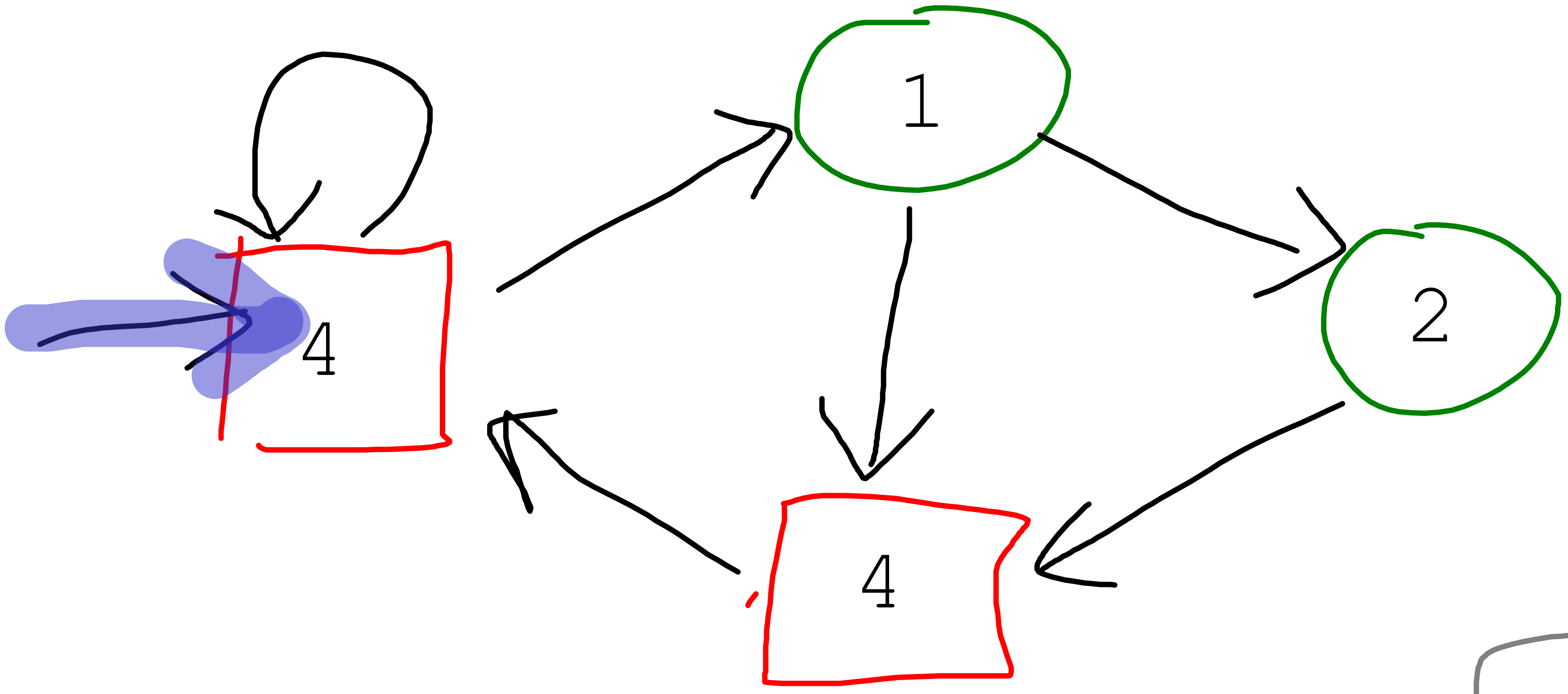
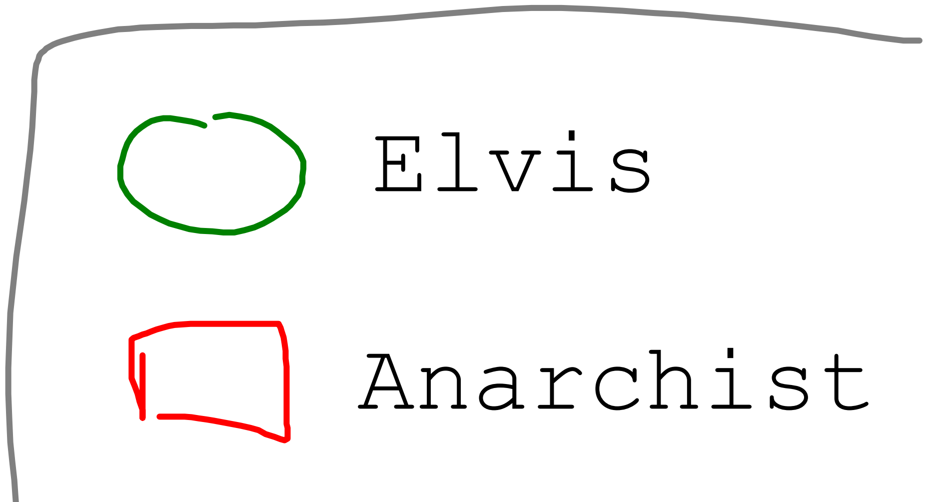


Play: 4



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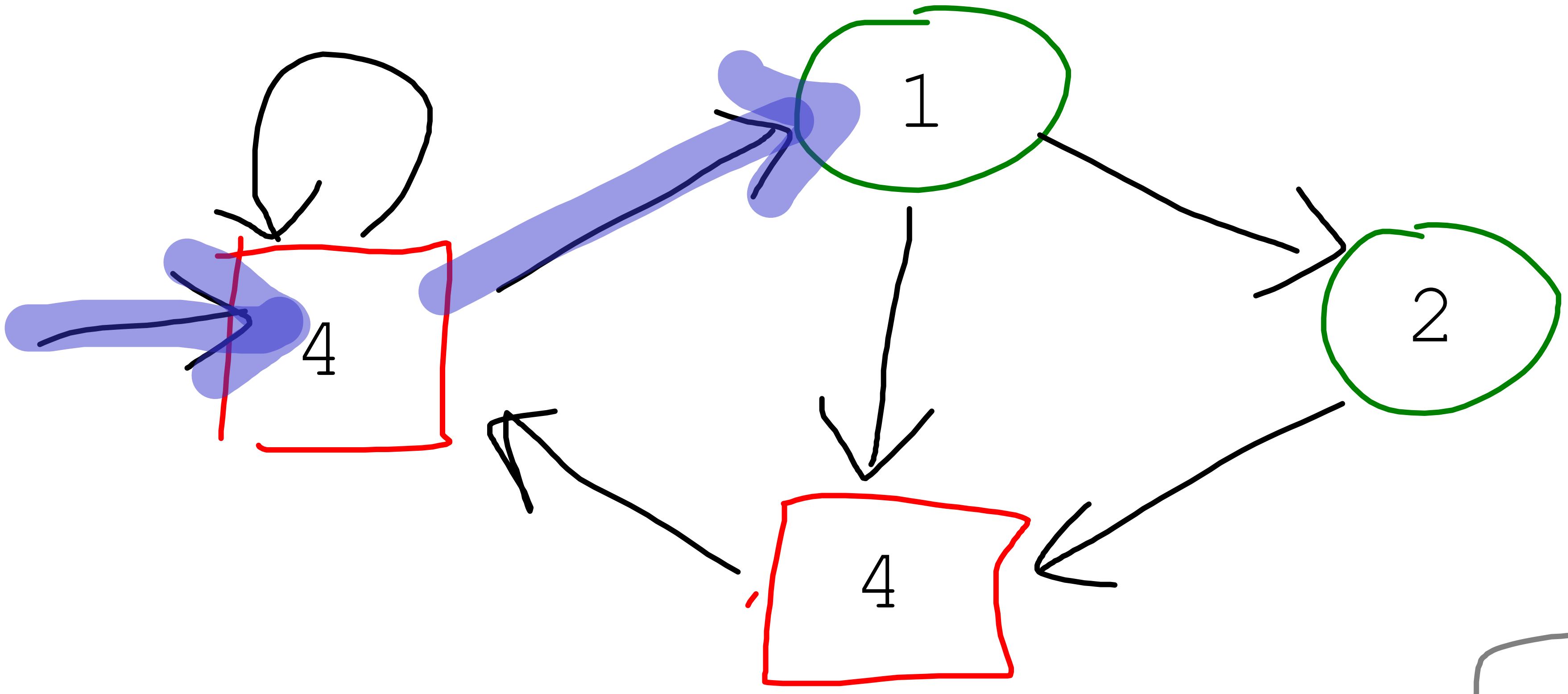
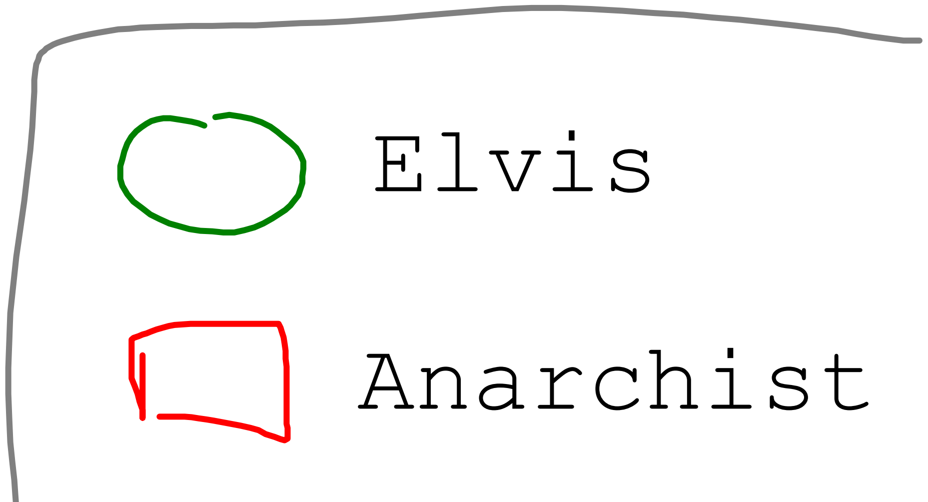


Play: 4 1



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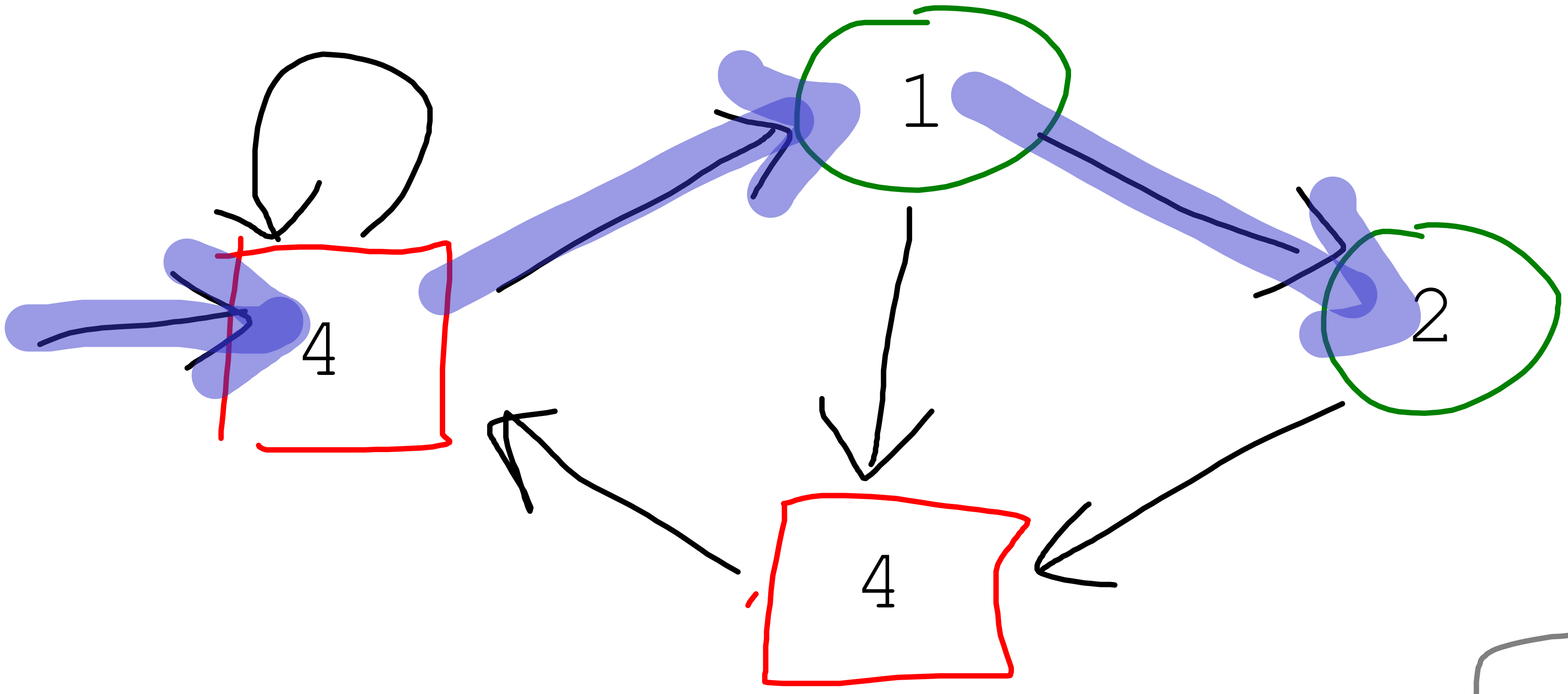
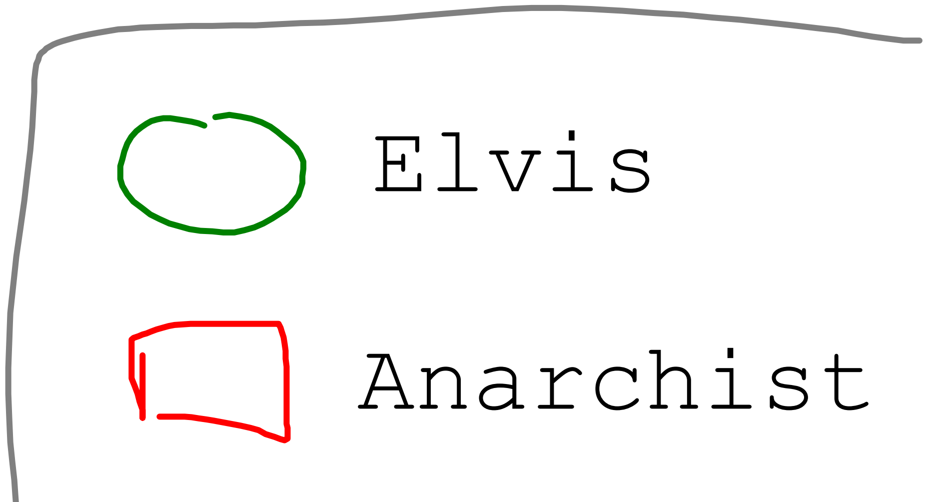


Play: 4 1 2



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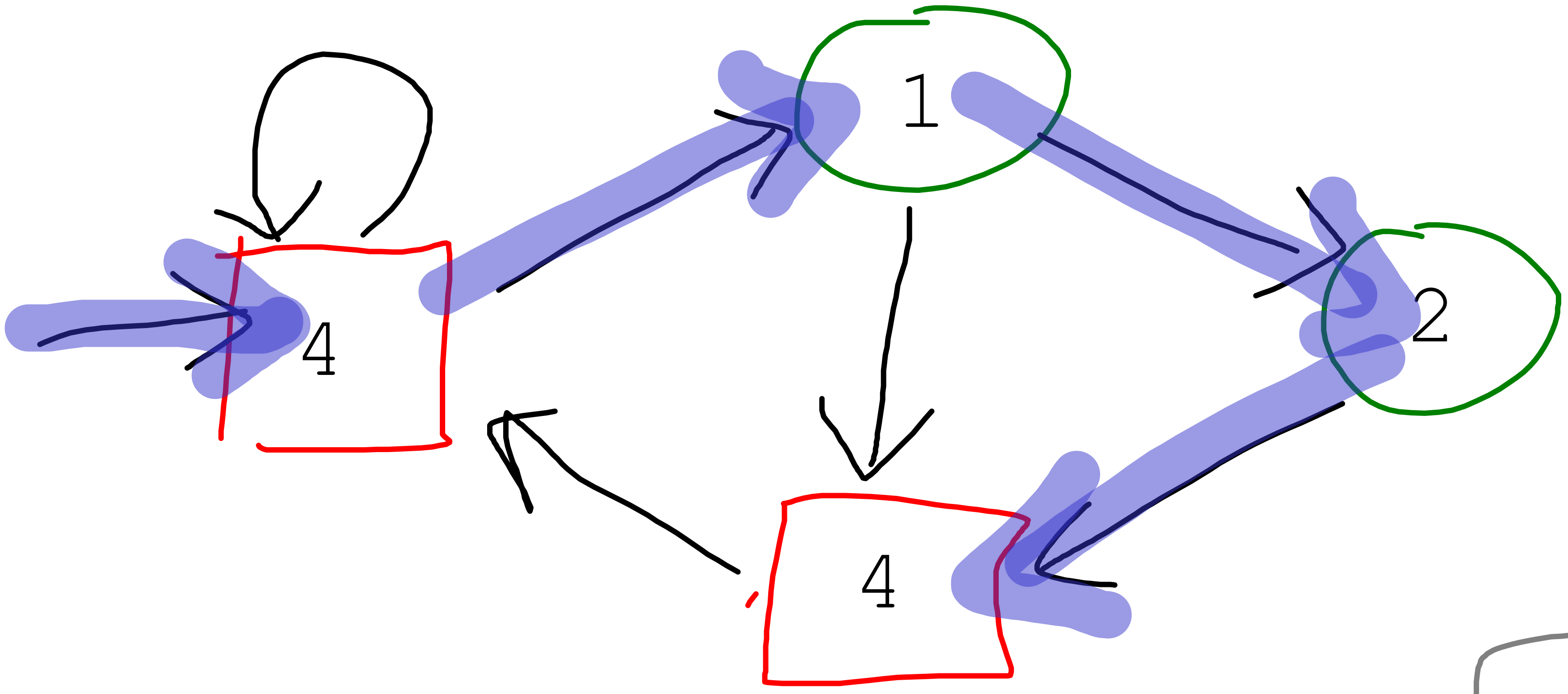
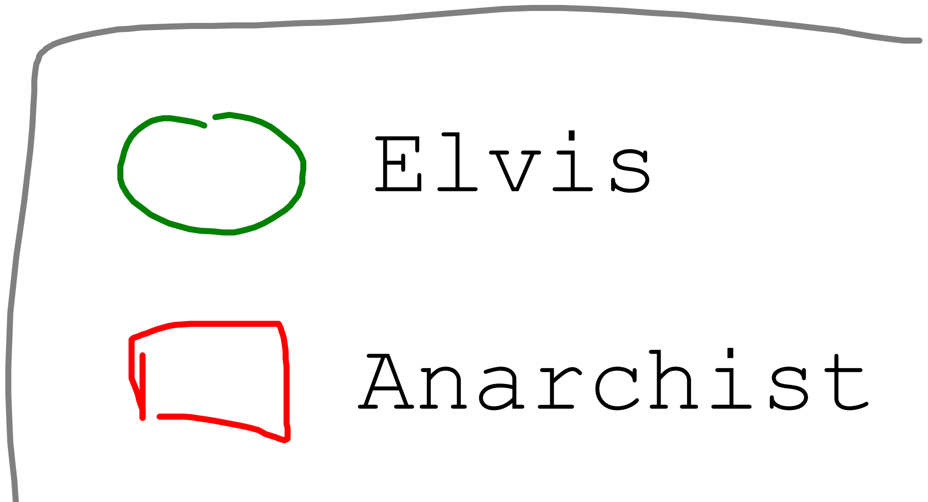


Play: 4 1 2 4



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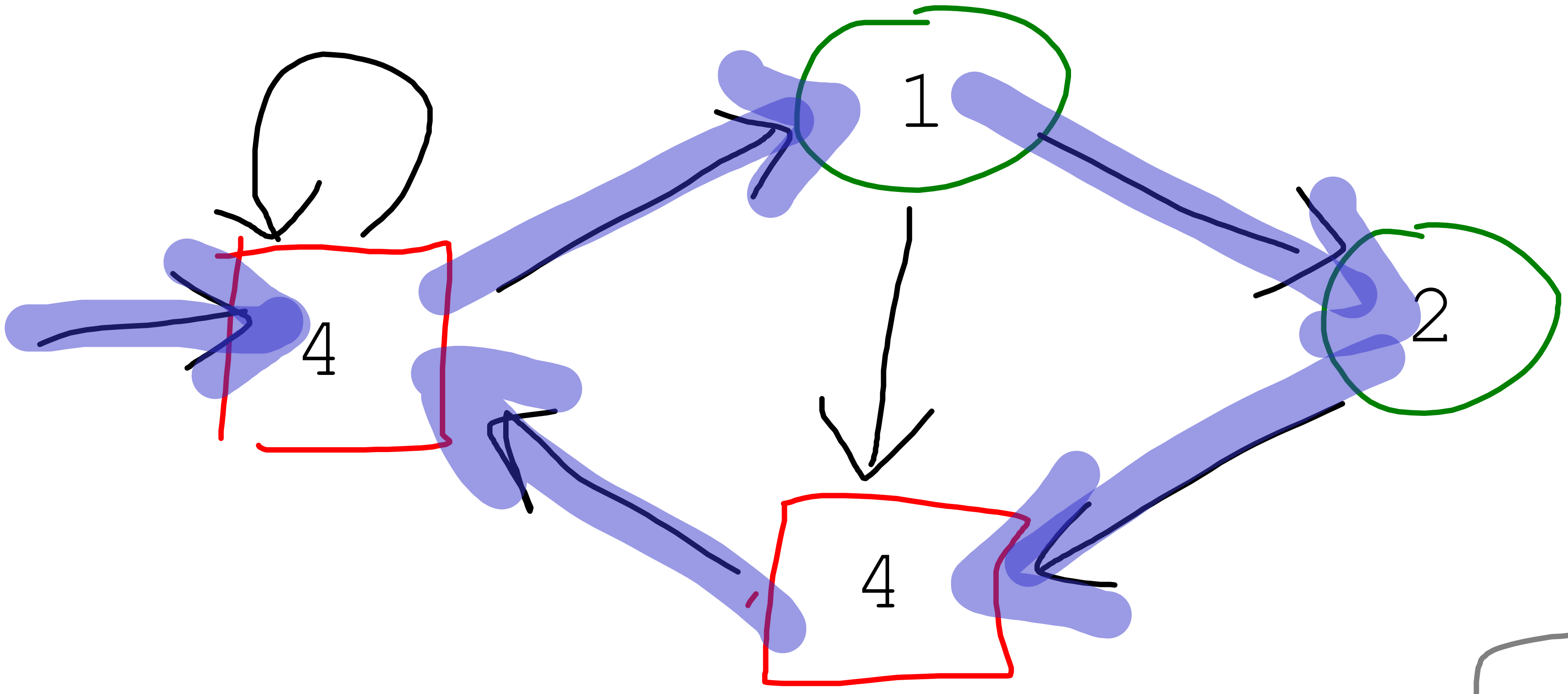

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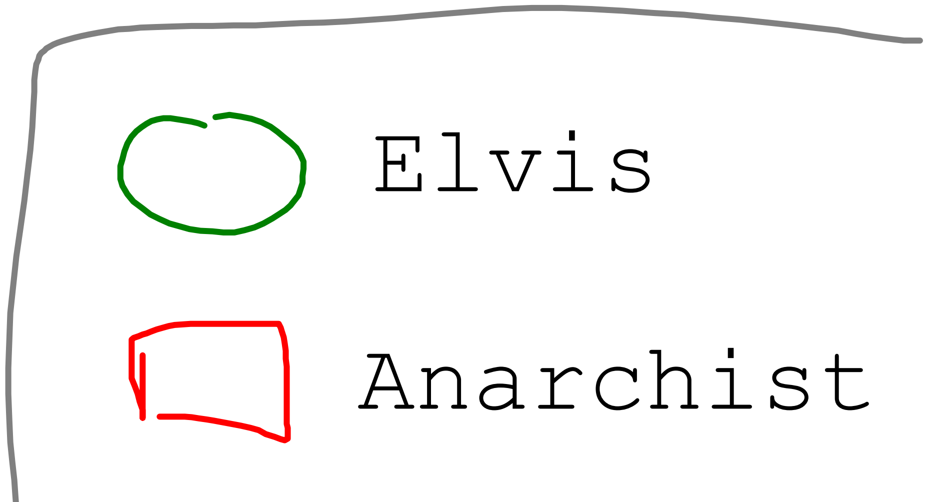

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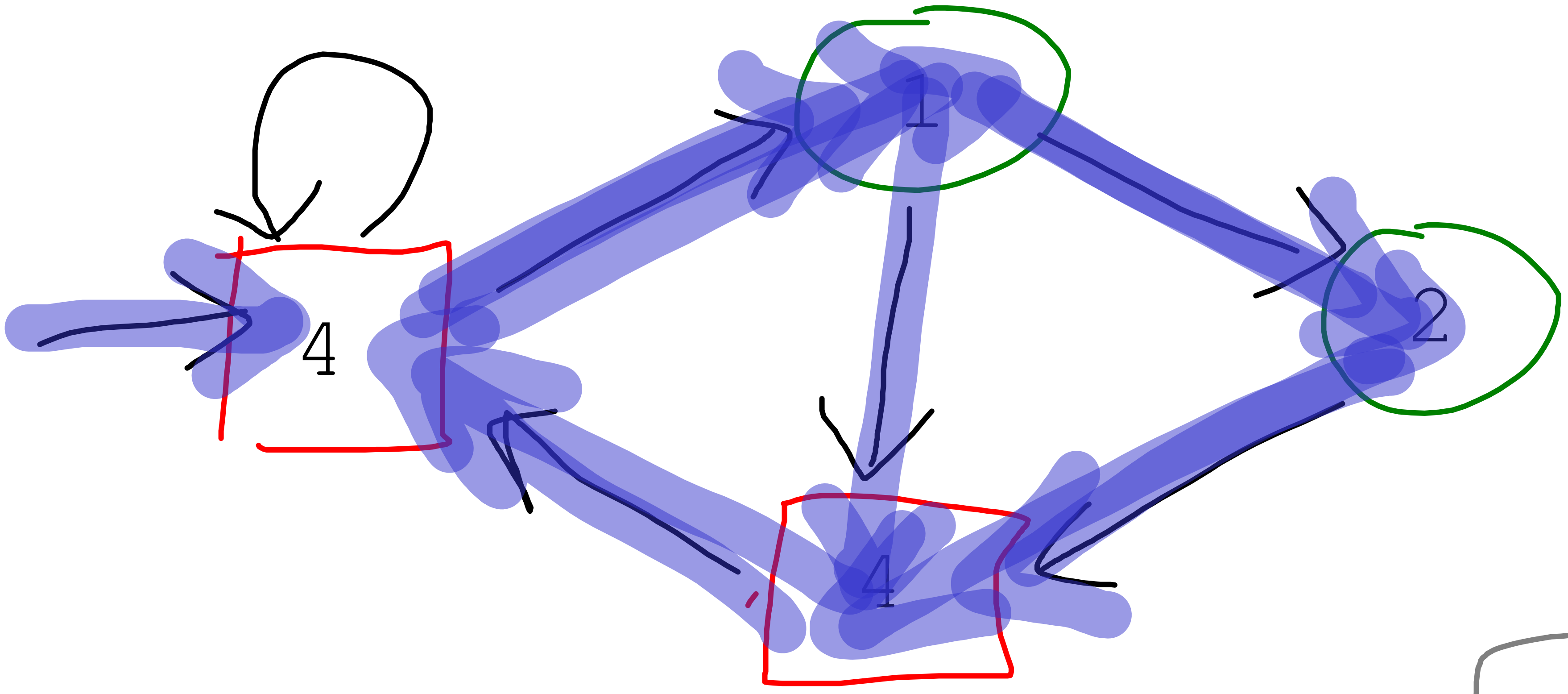
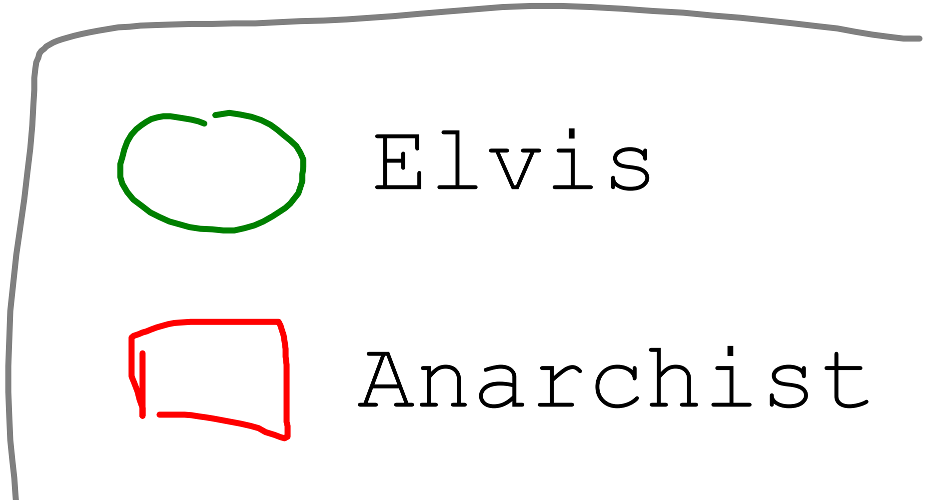


Play: 4 1 2 4 4 1 4 4 1 2 4...



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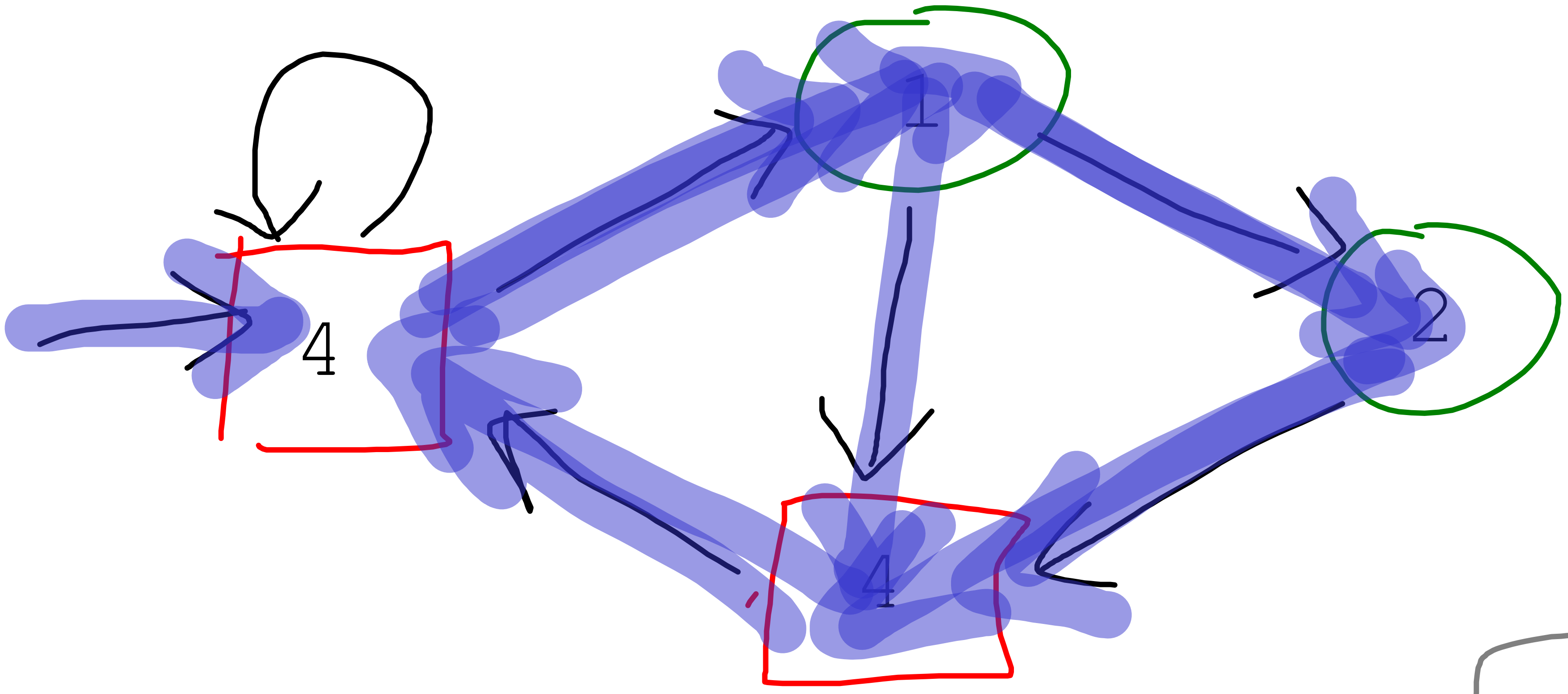
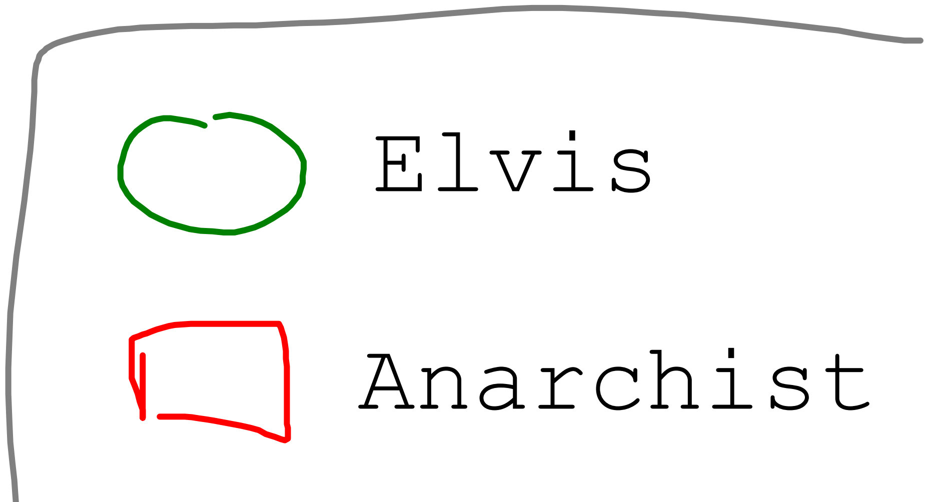
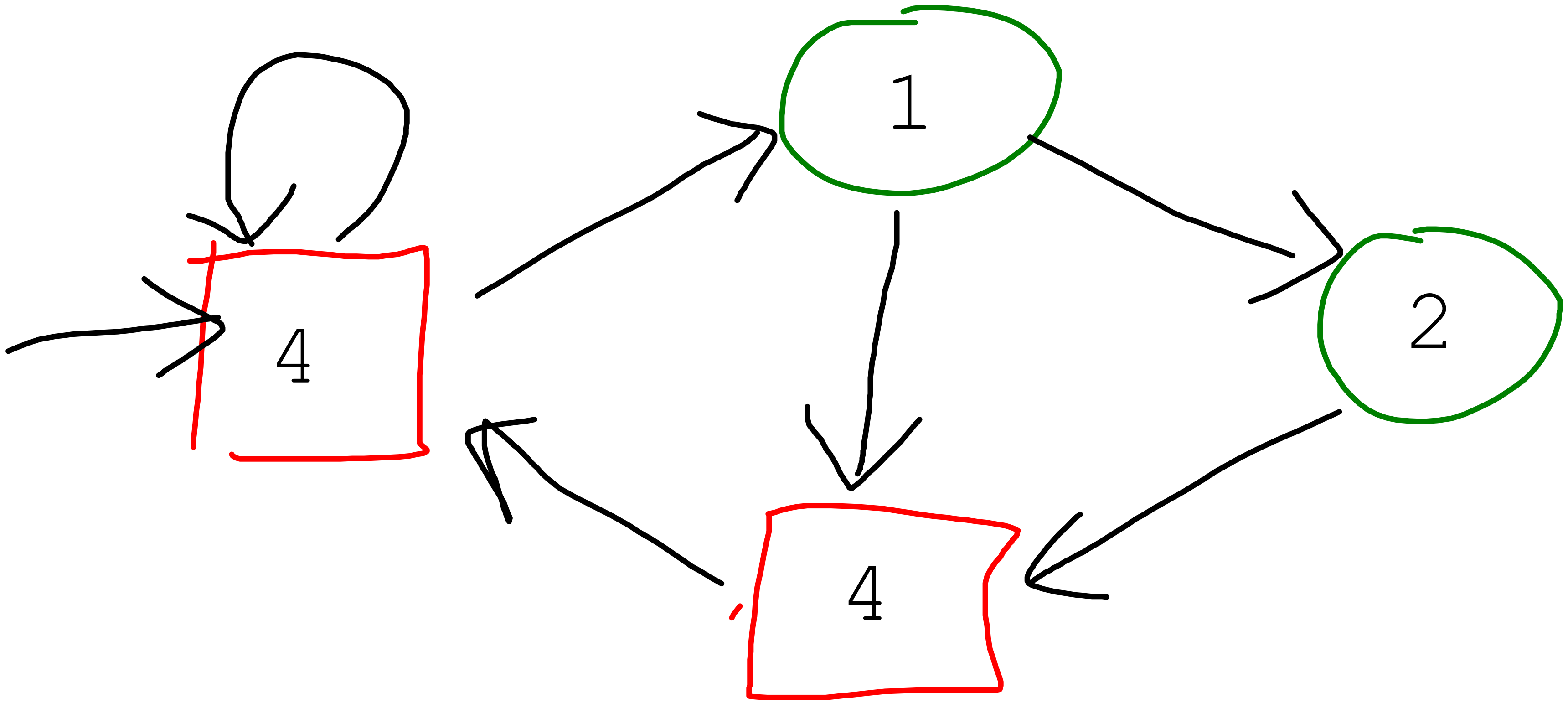


Play: 4 1 2 4 4 1 4 4 1 2 4...

Winner: Elvis if least infinitely occuring rank is even



Parity with Counters



Count: number of each rank
(without seeing anything smaller)

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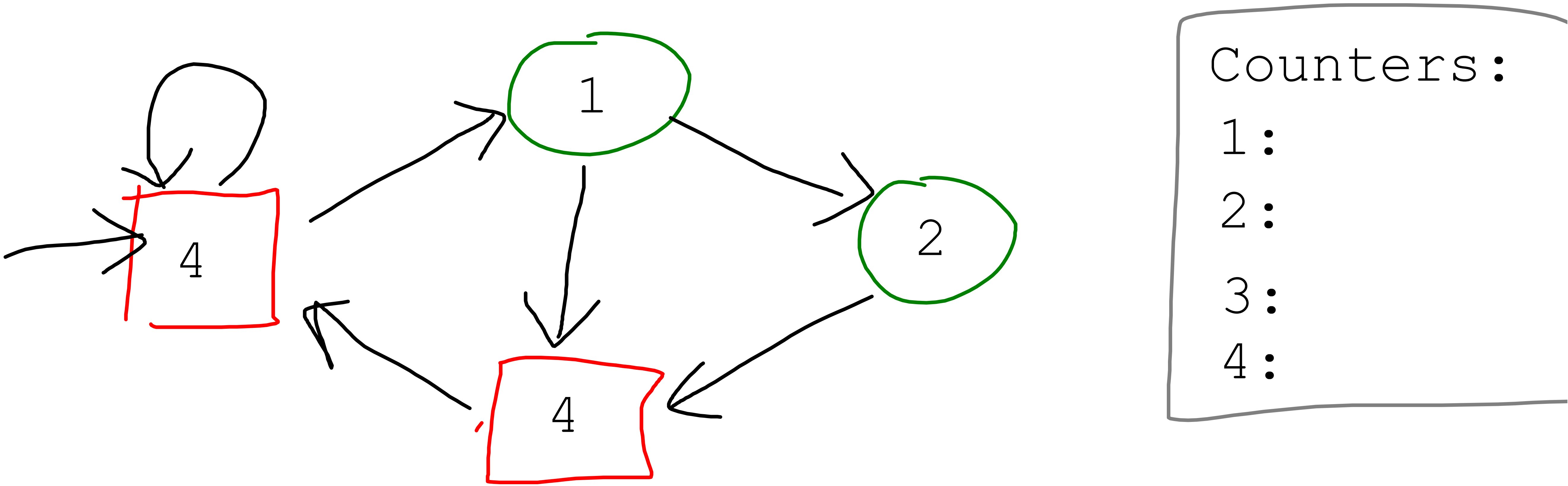

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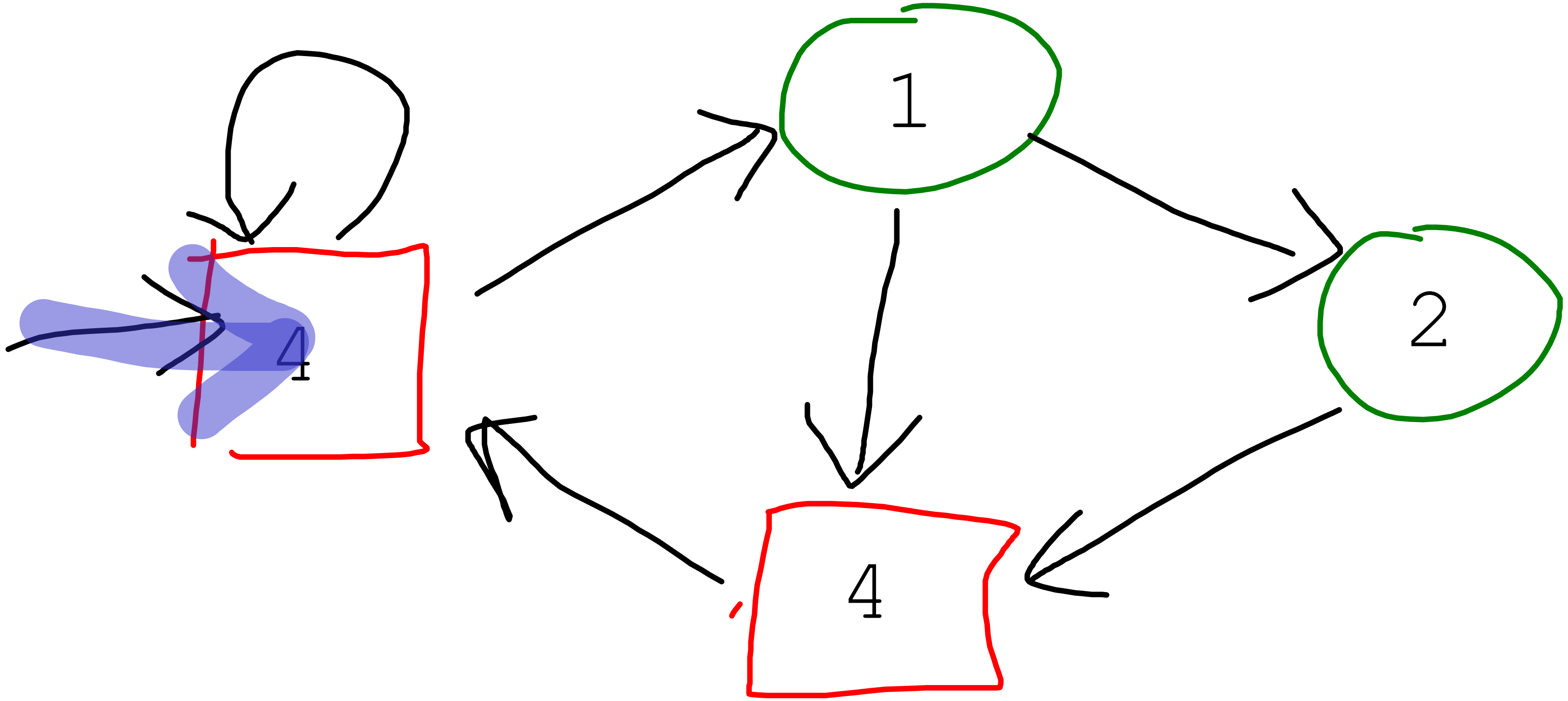

Counters:

1:

2:

3:

4: |

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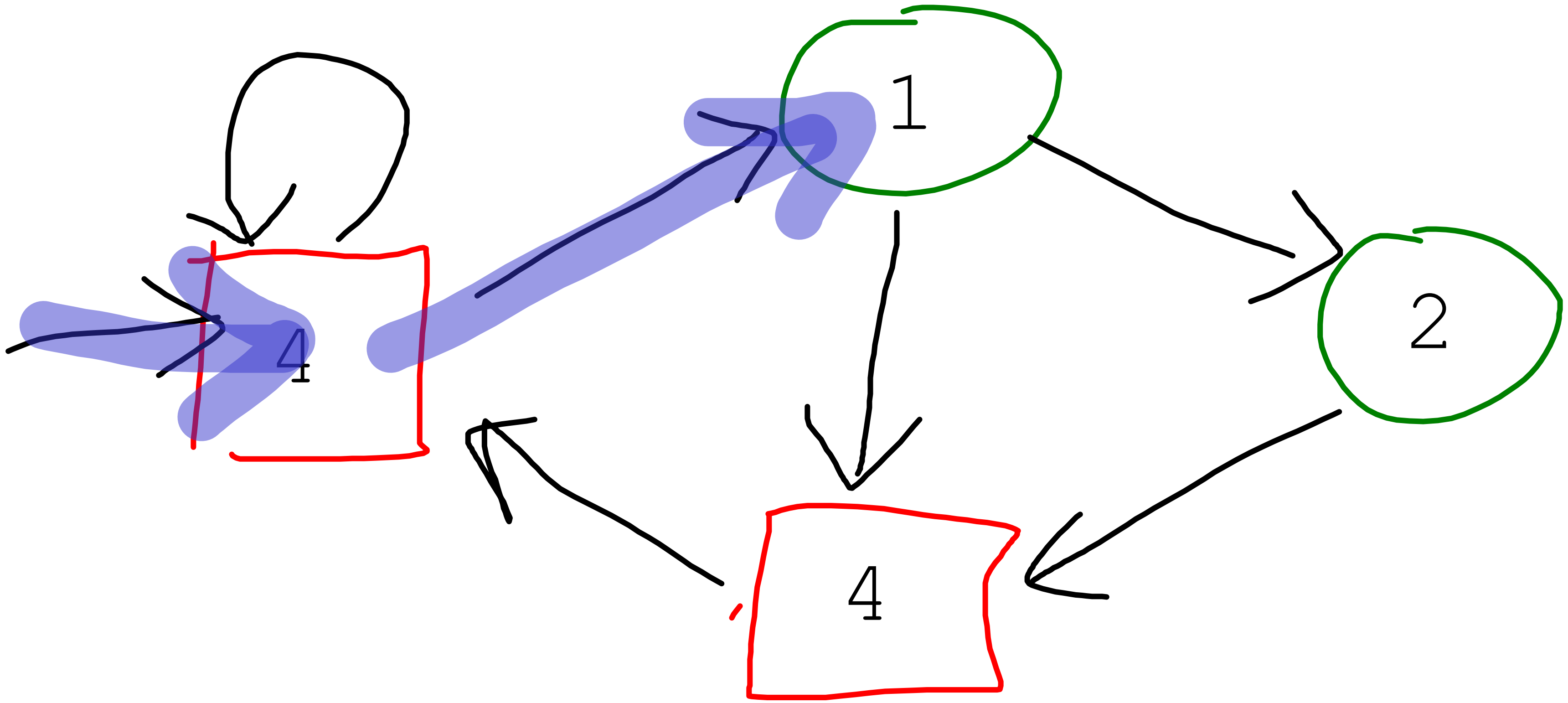

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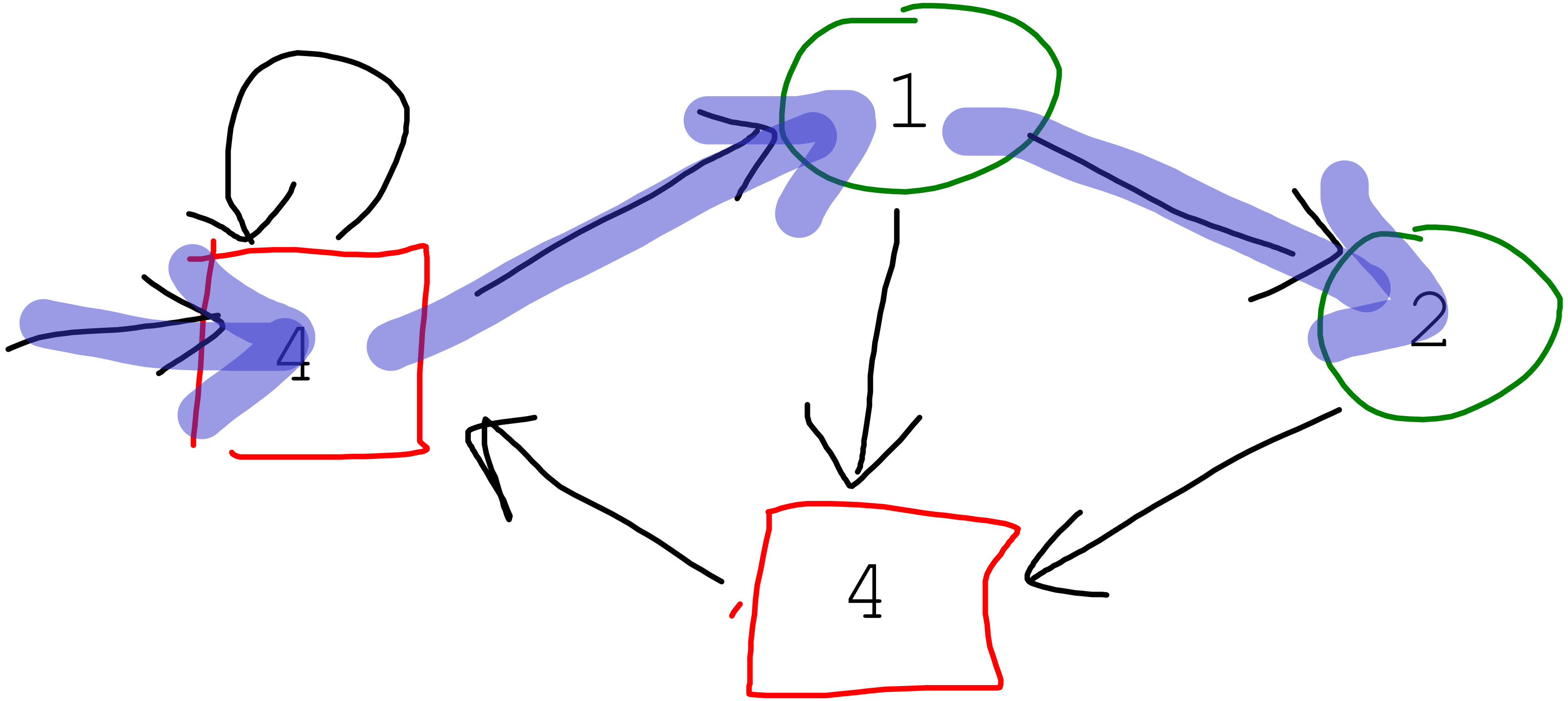

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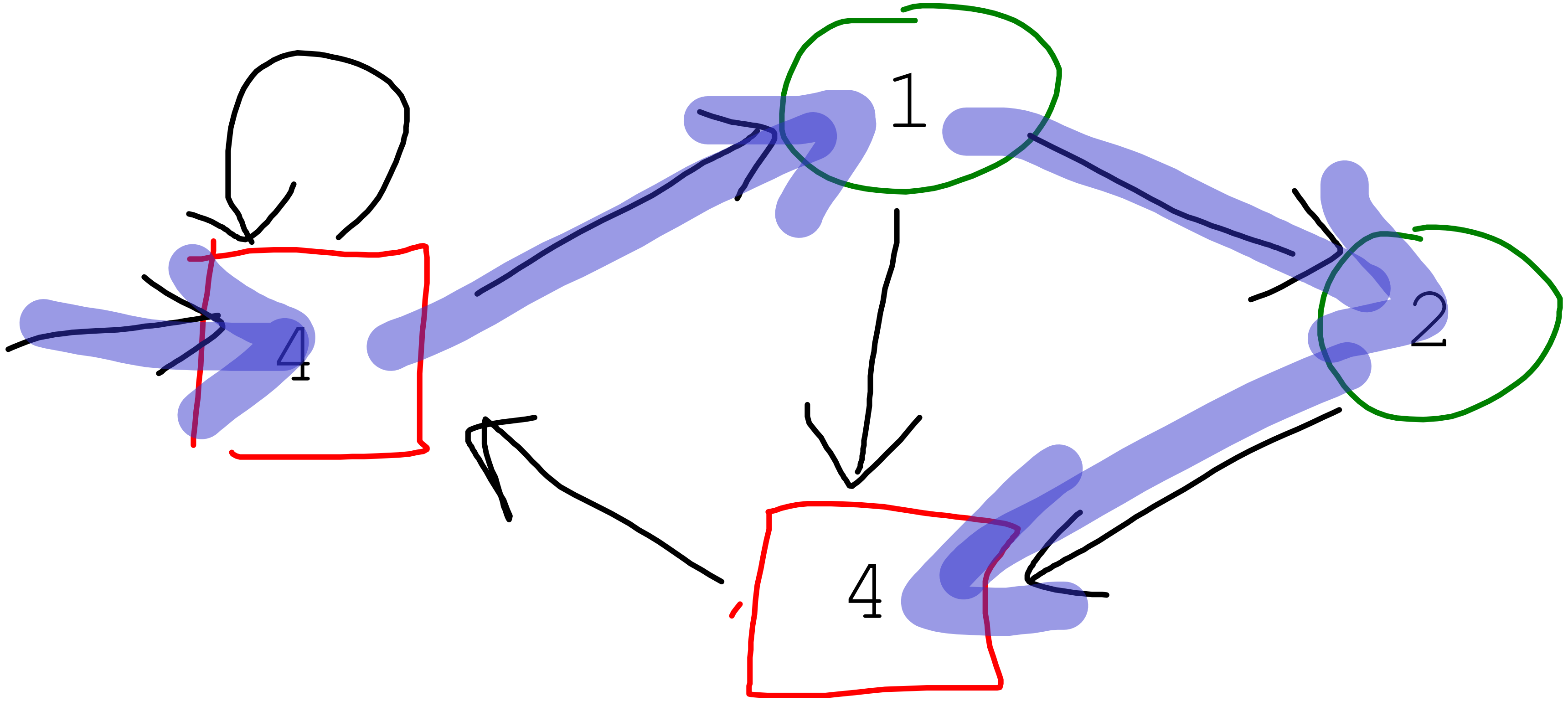

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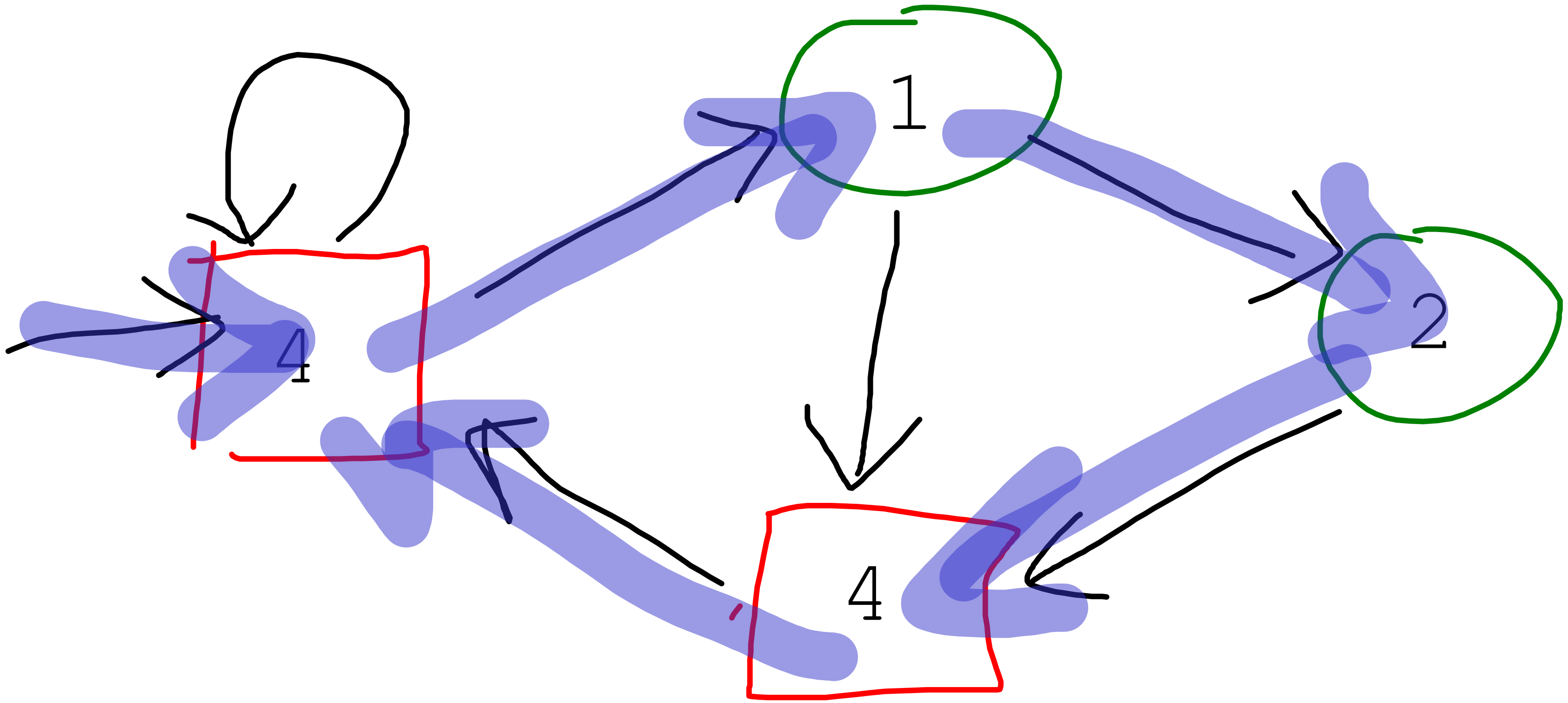

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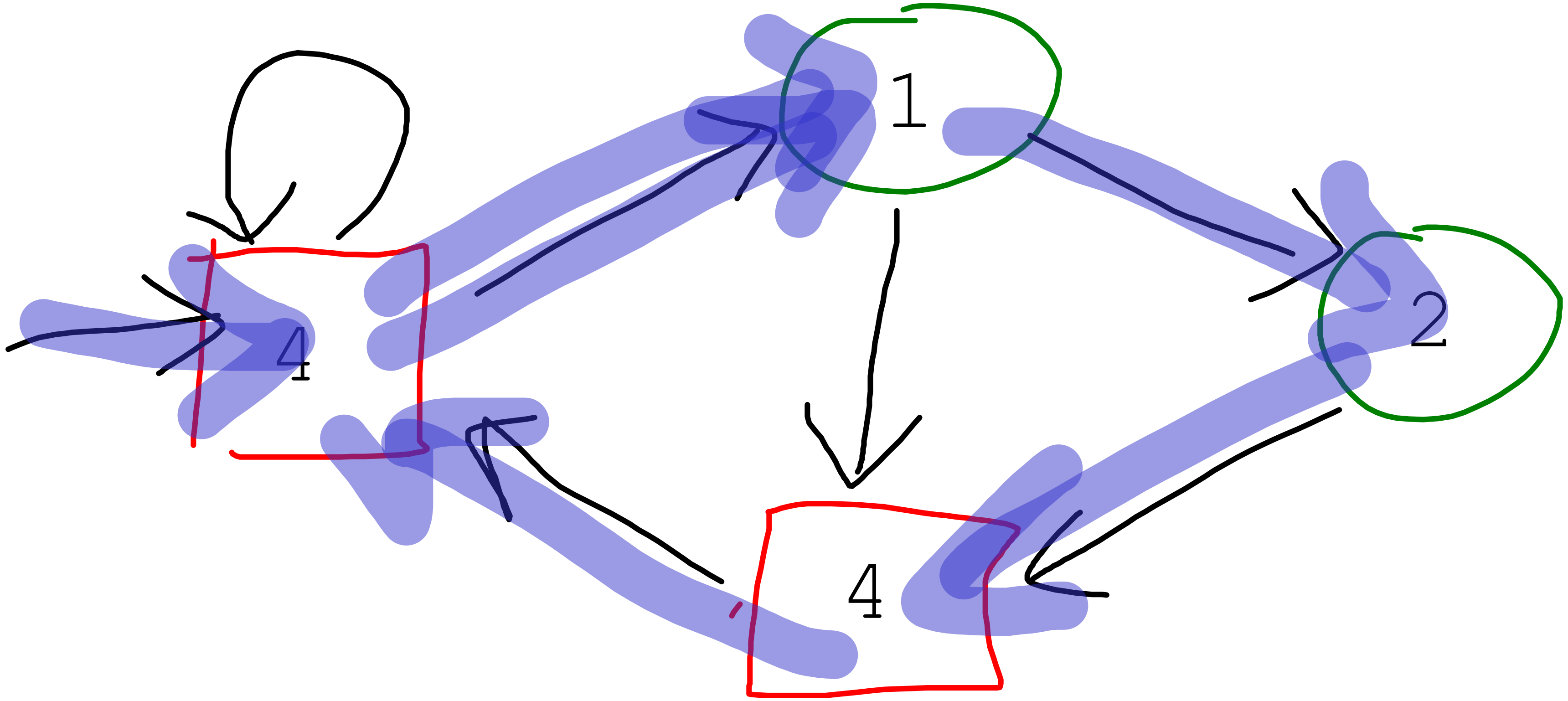

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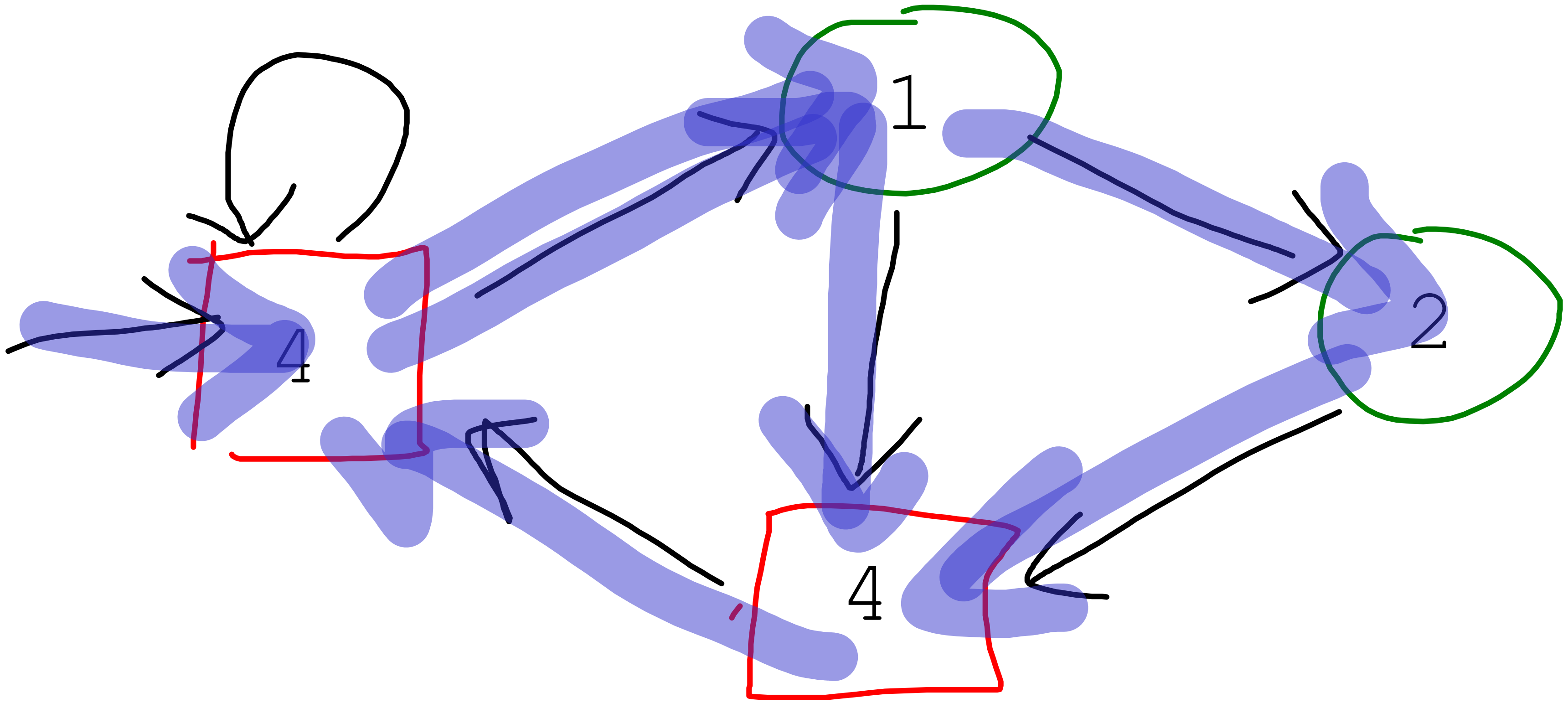

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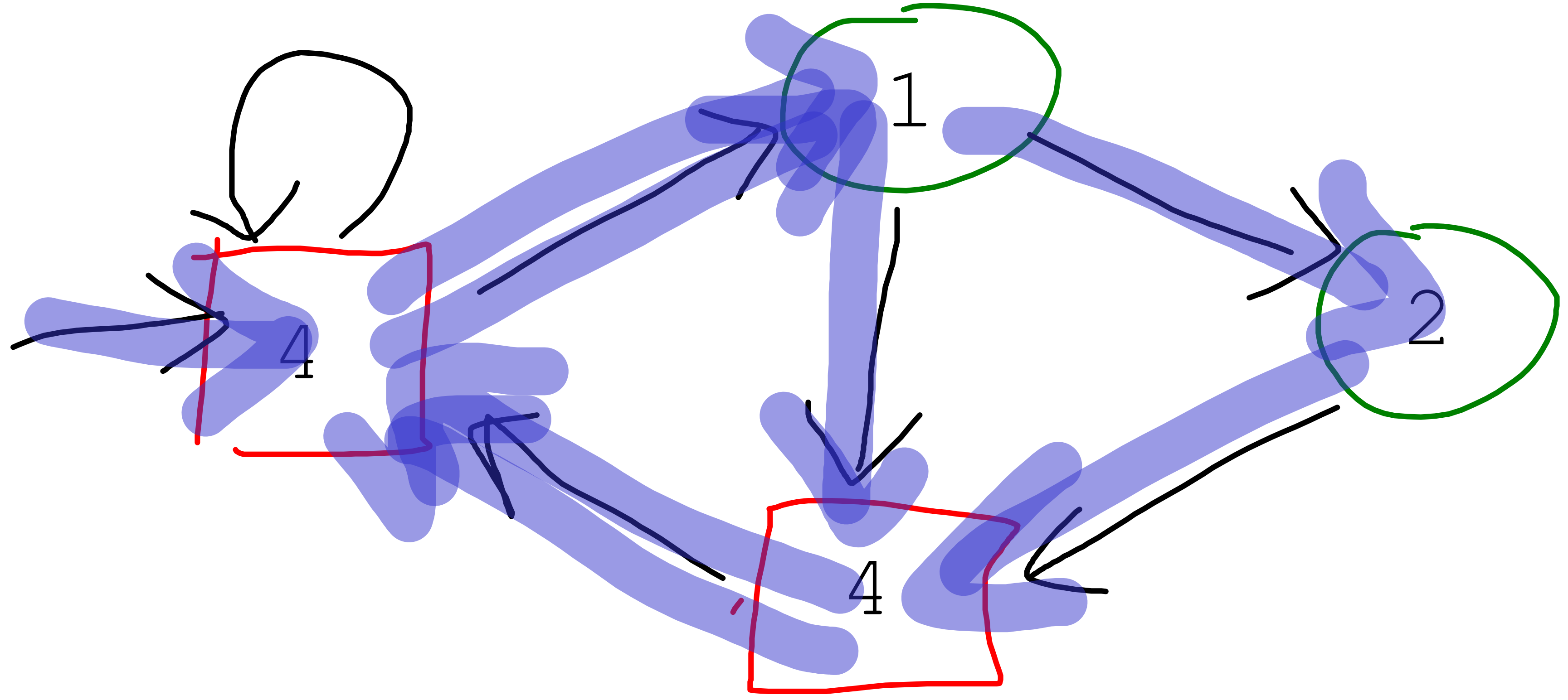

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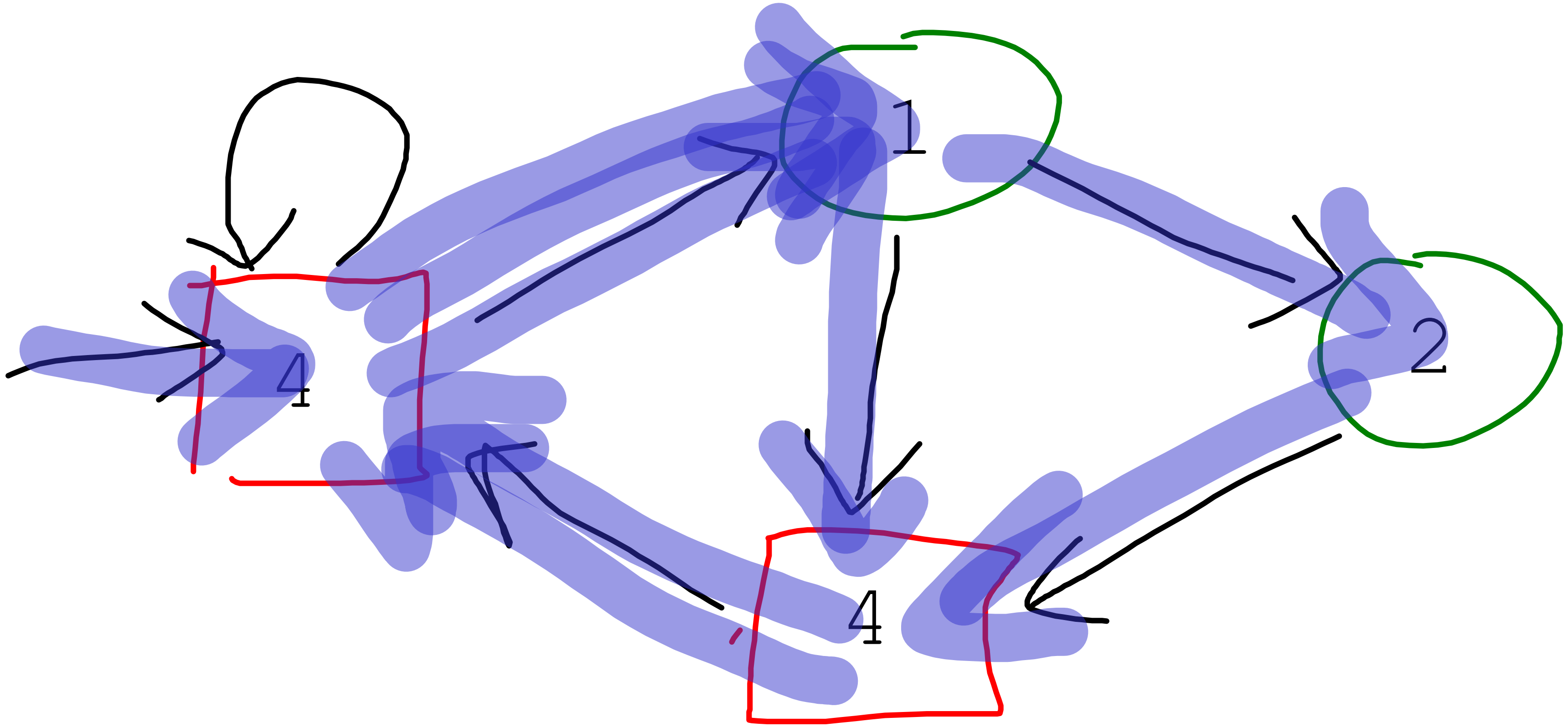

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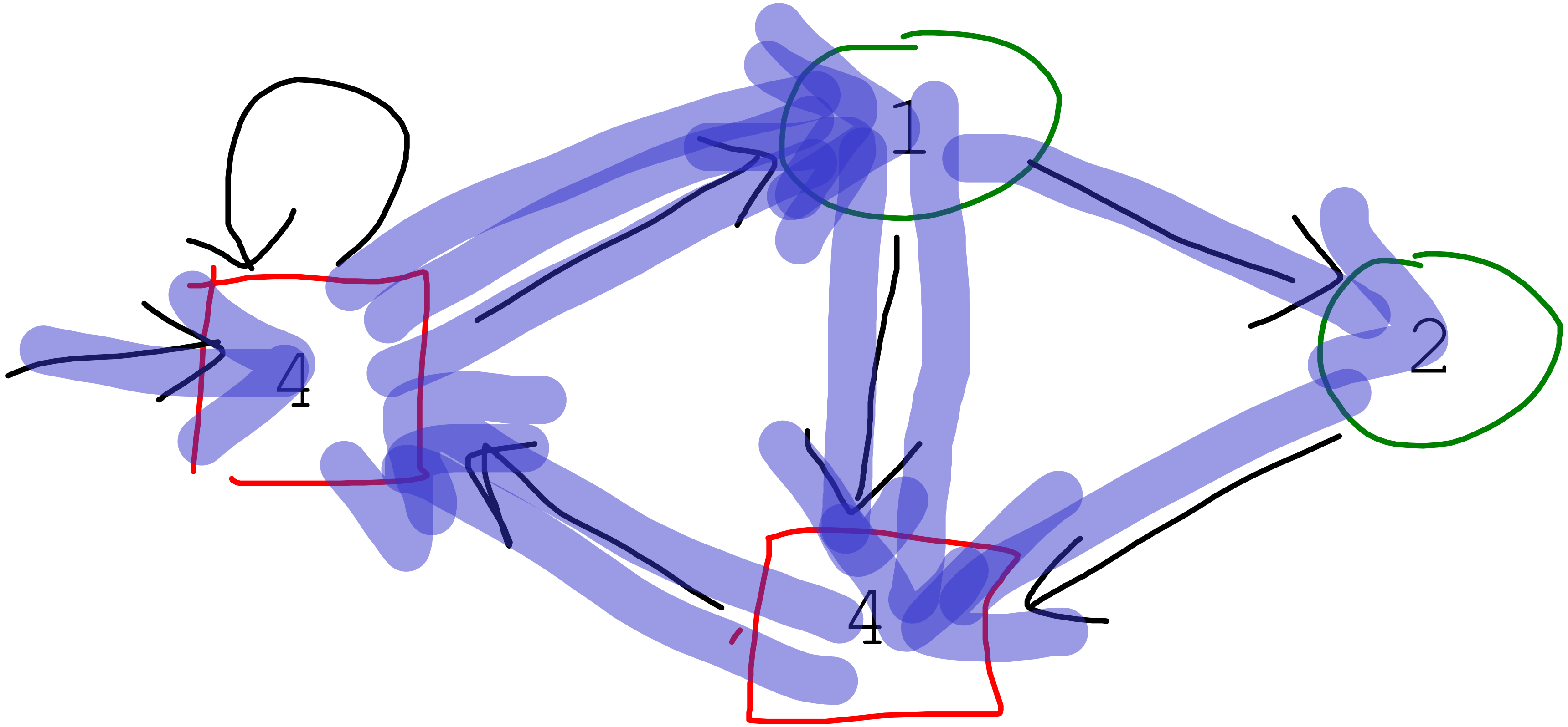

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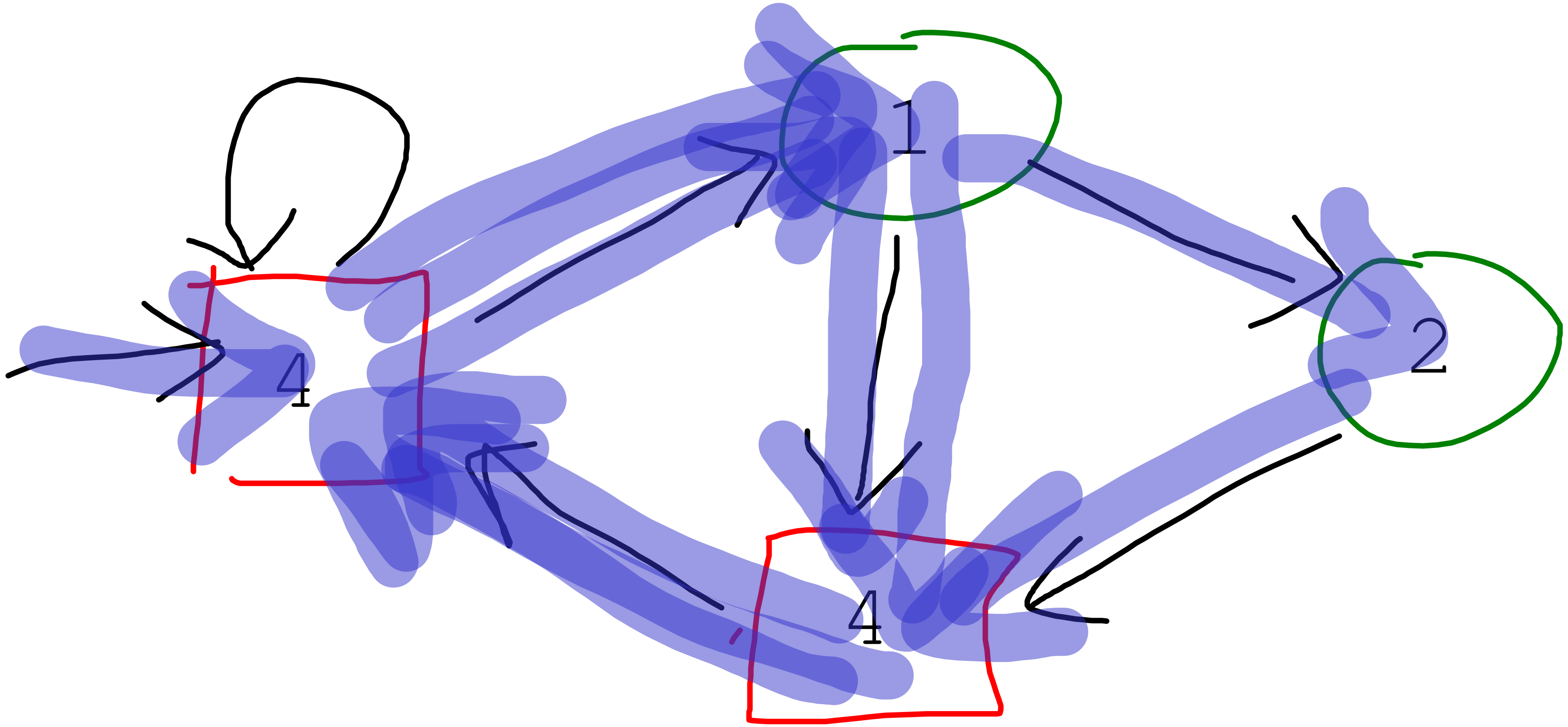

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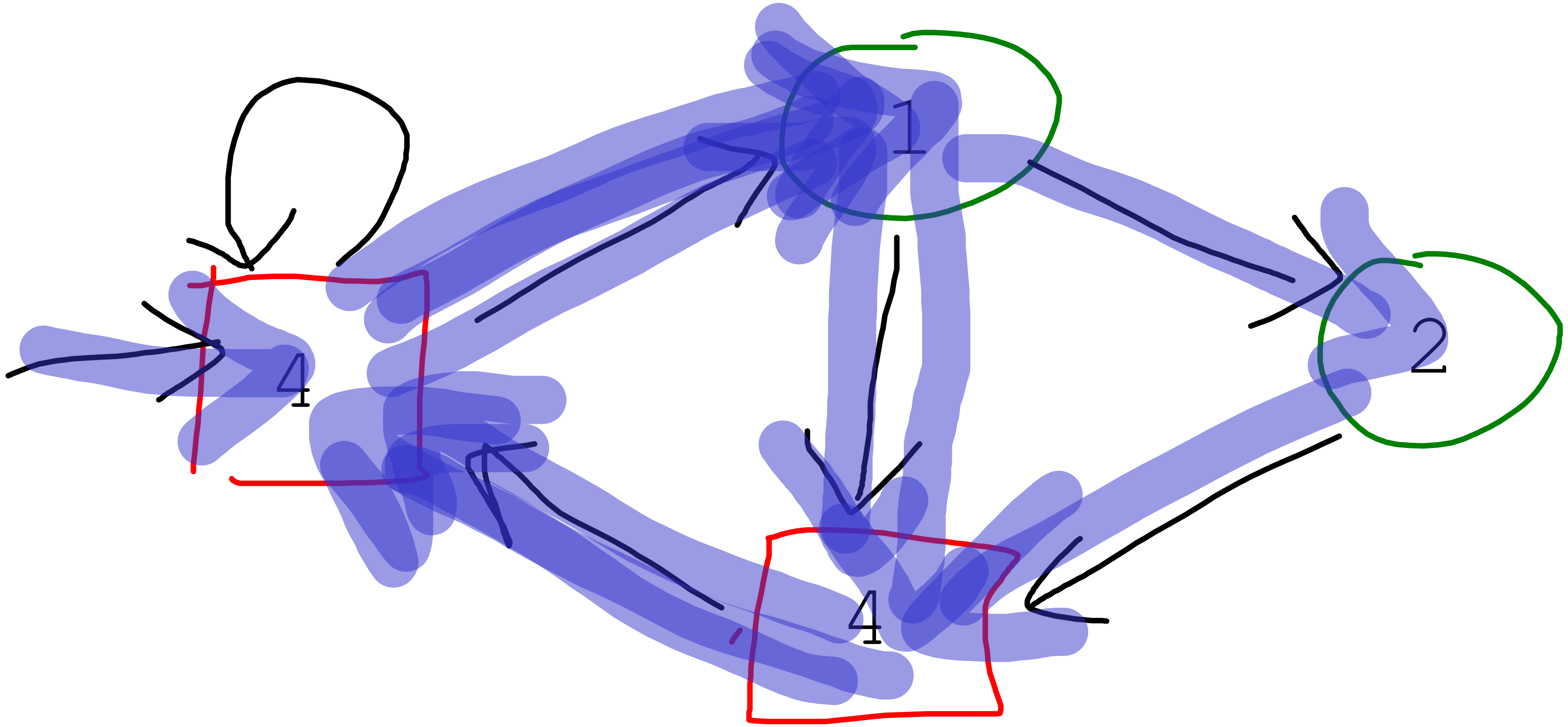

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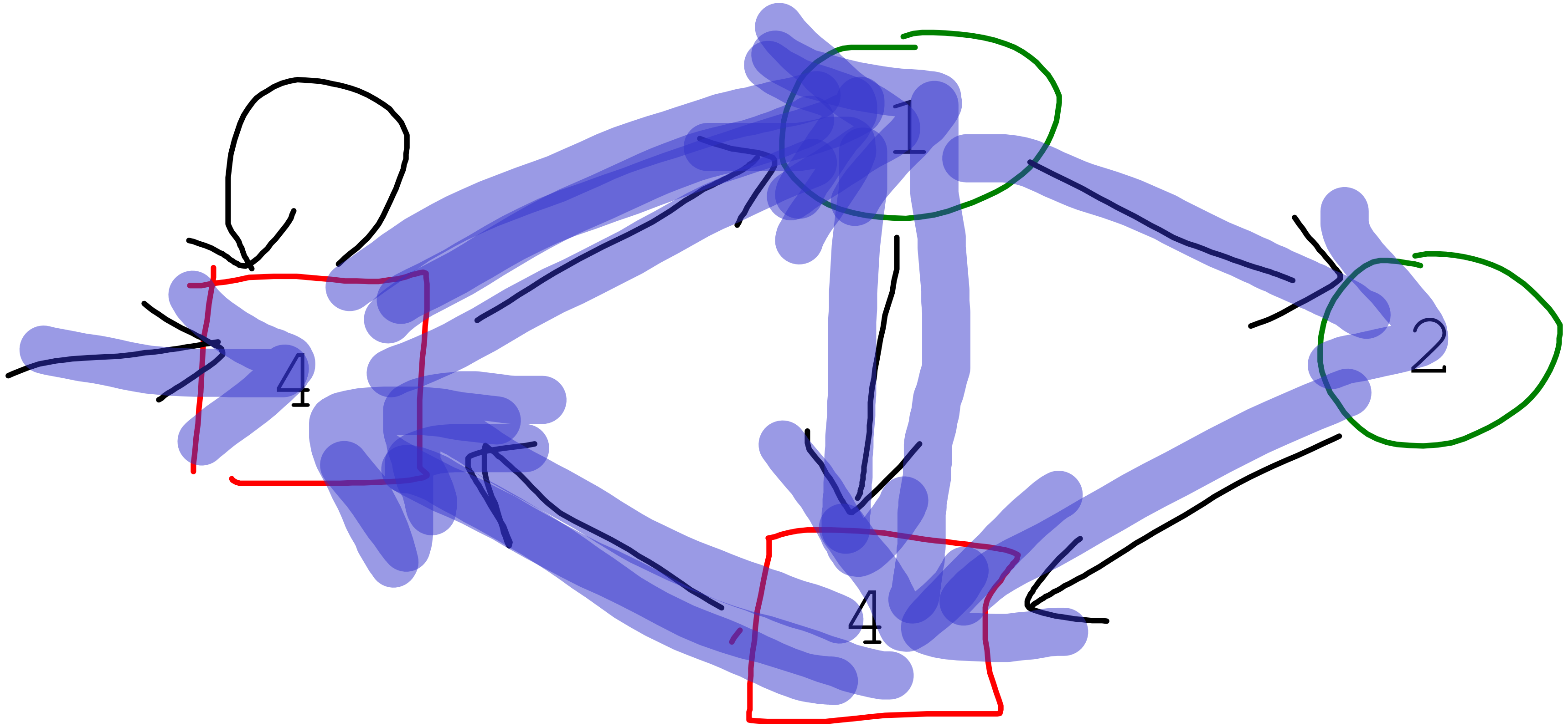

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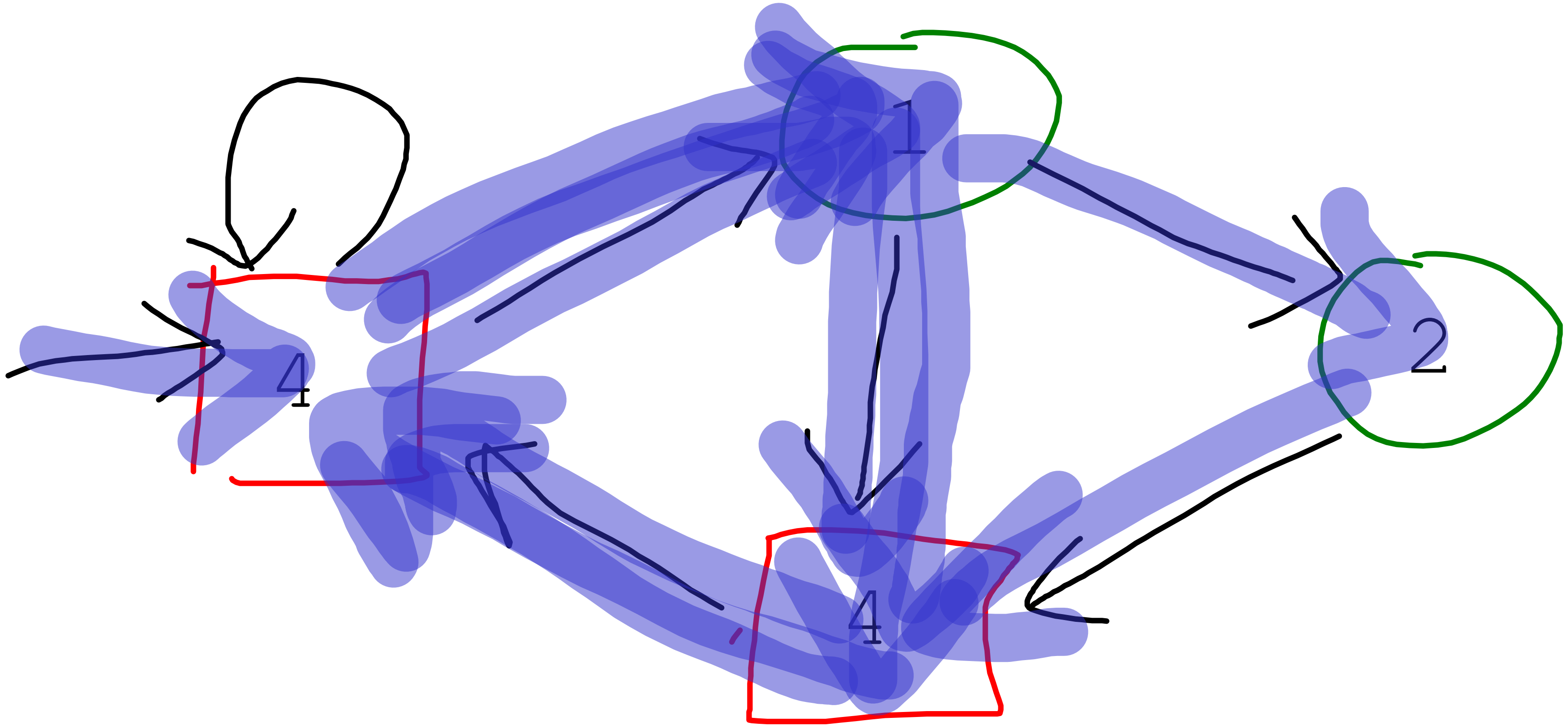

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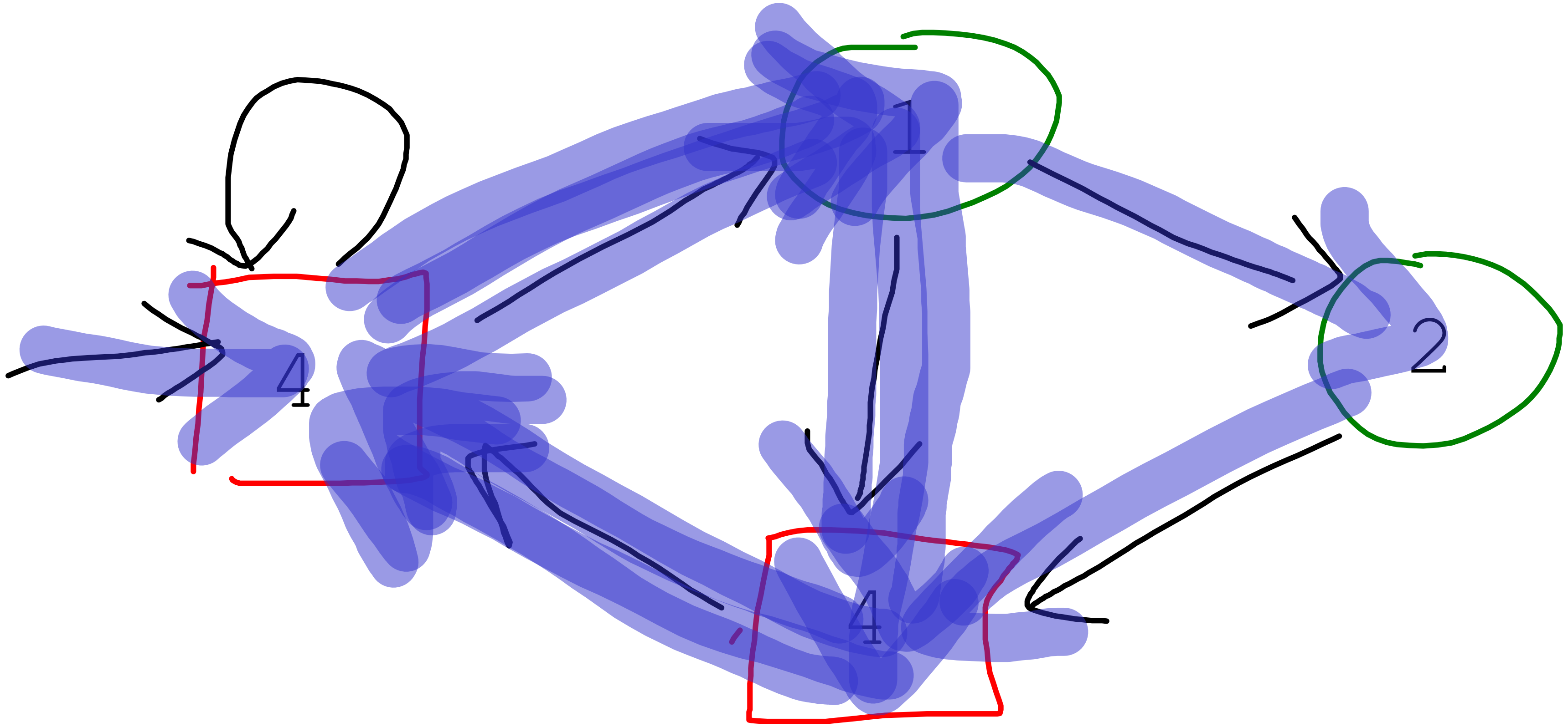

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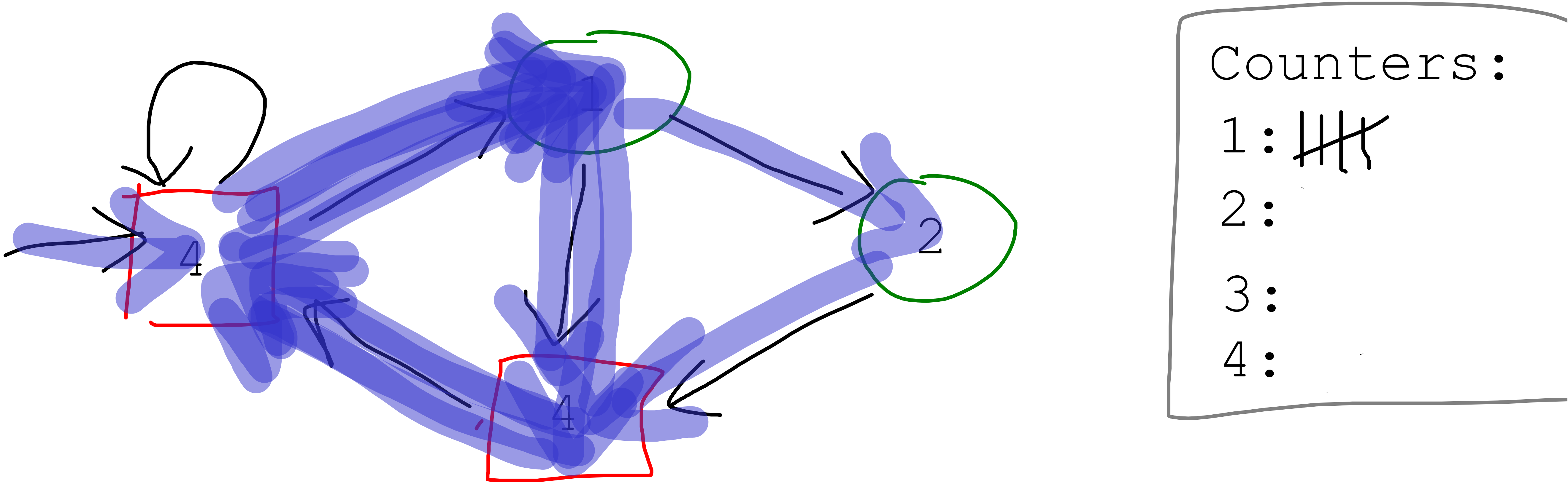

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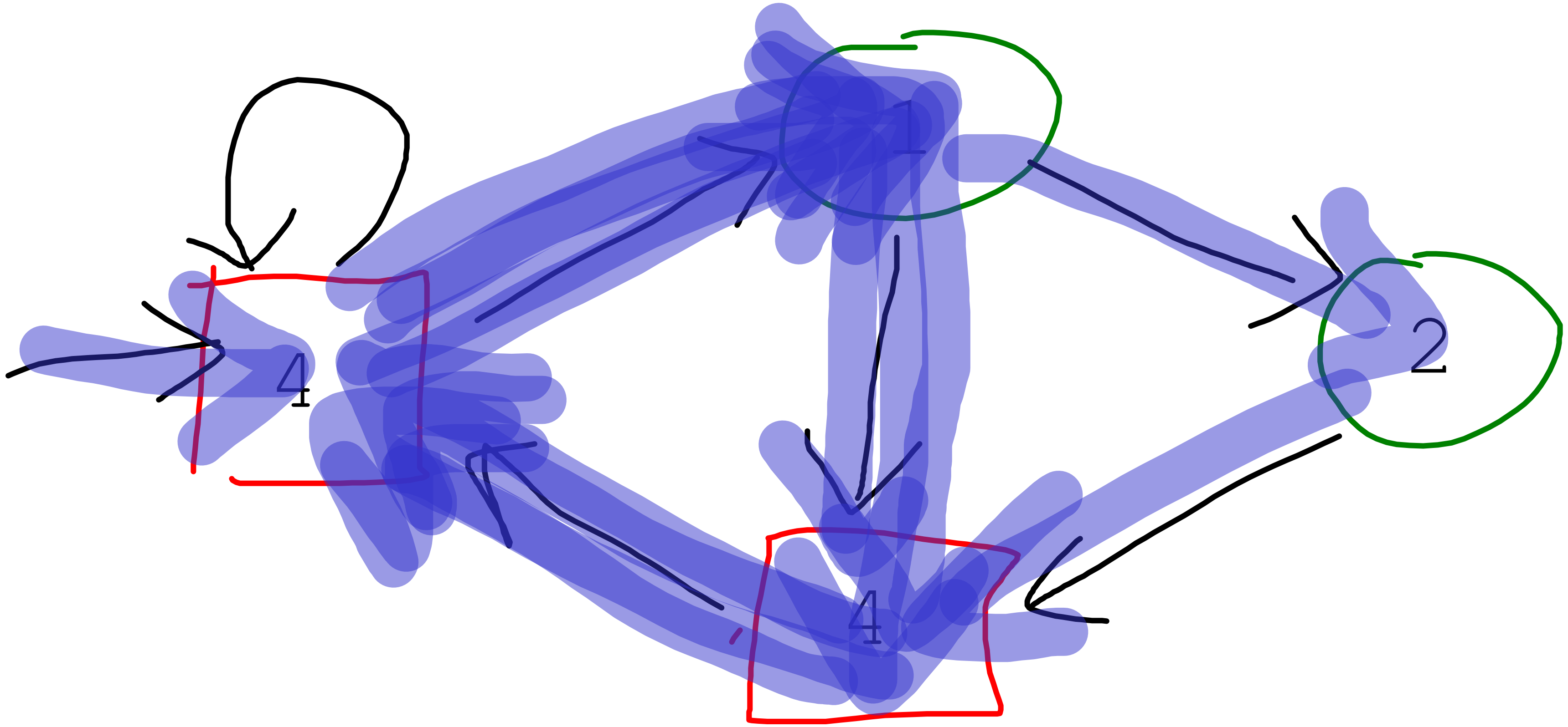
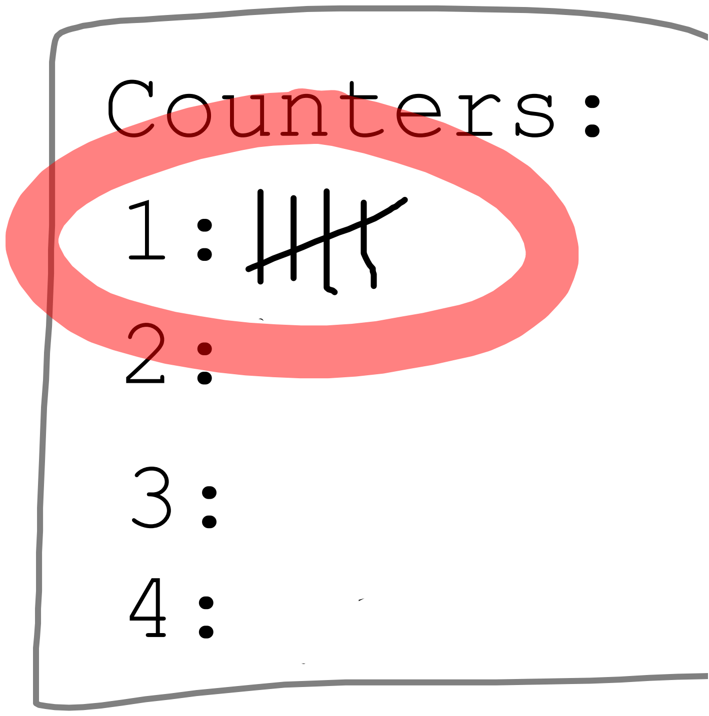
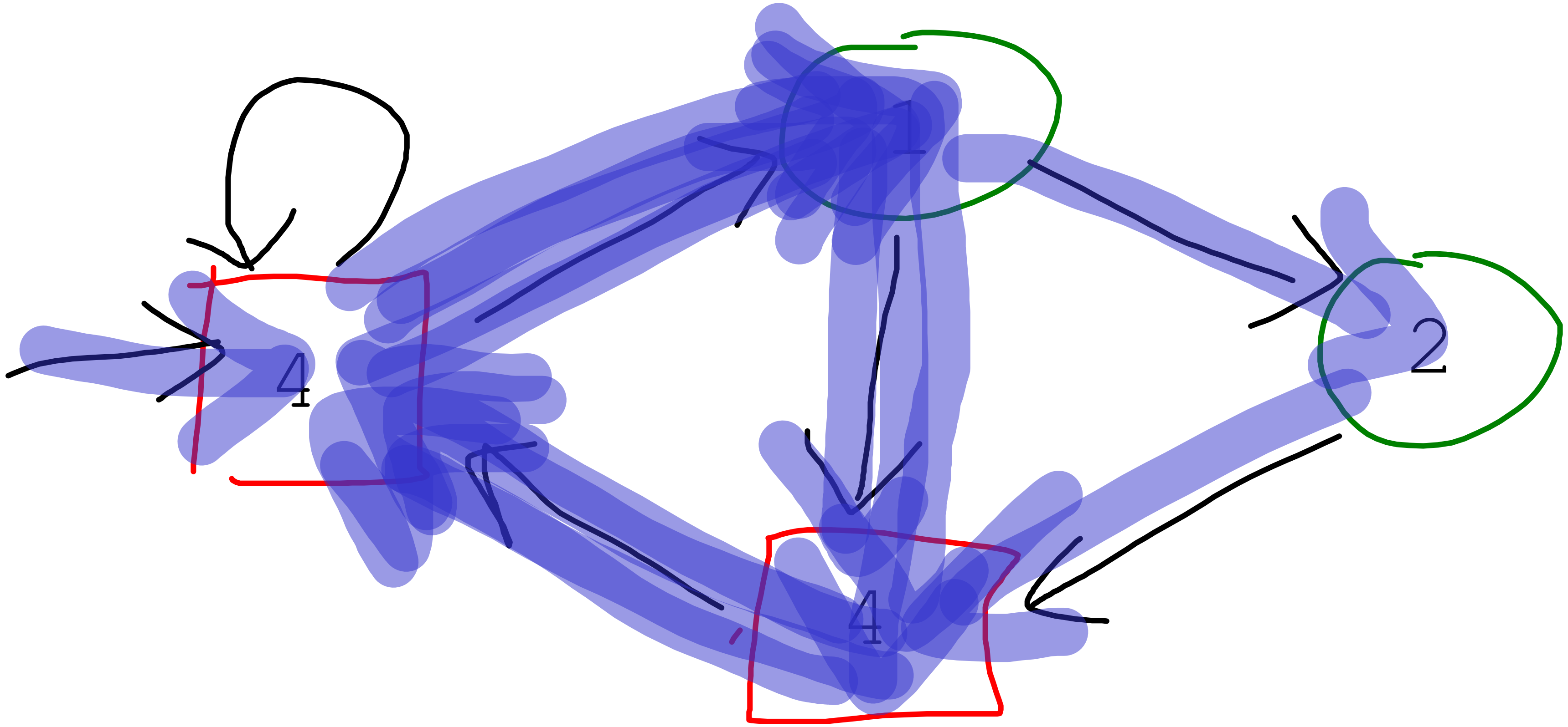
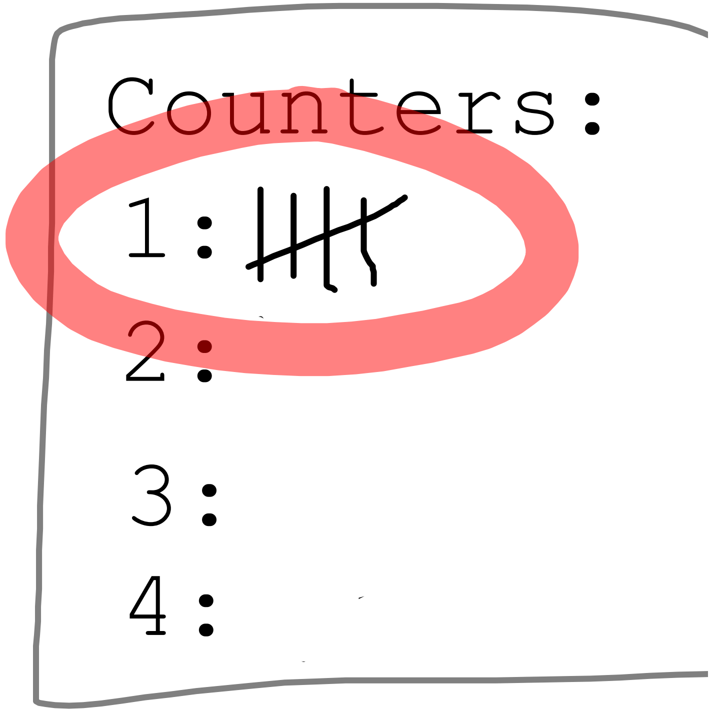




There is a loop!
(Hit 1 five times over 4 states)

Parity with Counters





There is a loop!
(Hit 1 five times over 4 states)

The smallest rank is odd: Elvis loses!

Parity to Safety

Reduce parity to safety game (Bernet et al):

- Keep counters in state (n * n^k states)
- Elvis loses if odd counter is high

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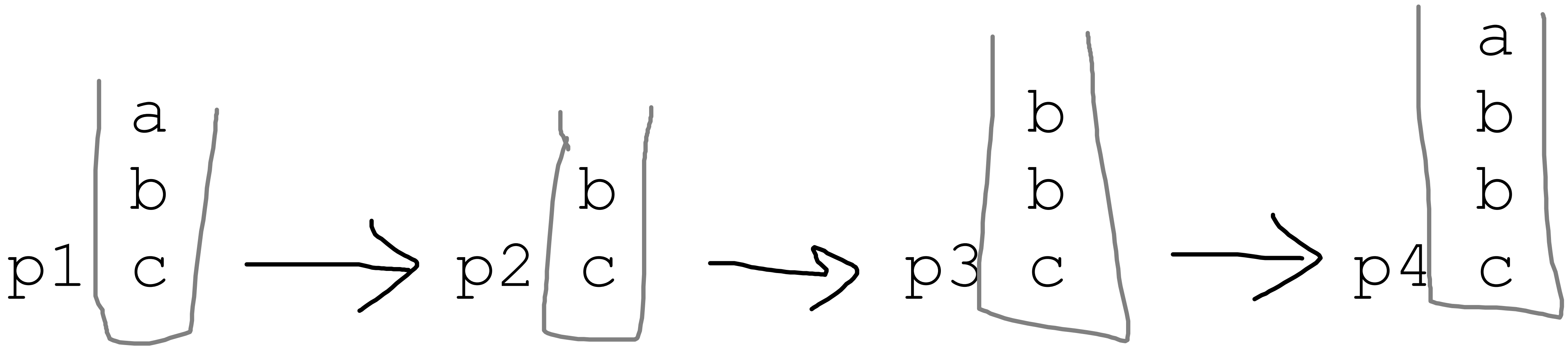
Exponential blow up!

Pushdown Games



Each state has:

Control state (from finite set)

Stack of characters

Model recursive programs

Pushdown Parity to Safety

Pushdown Parity

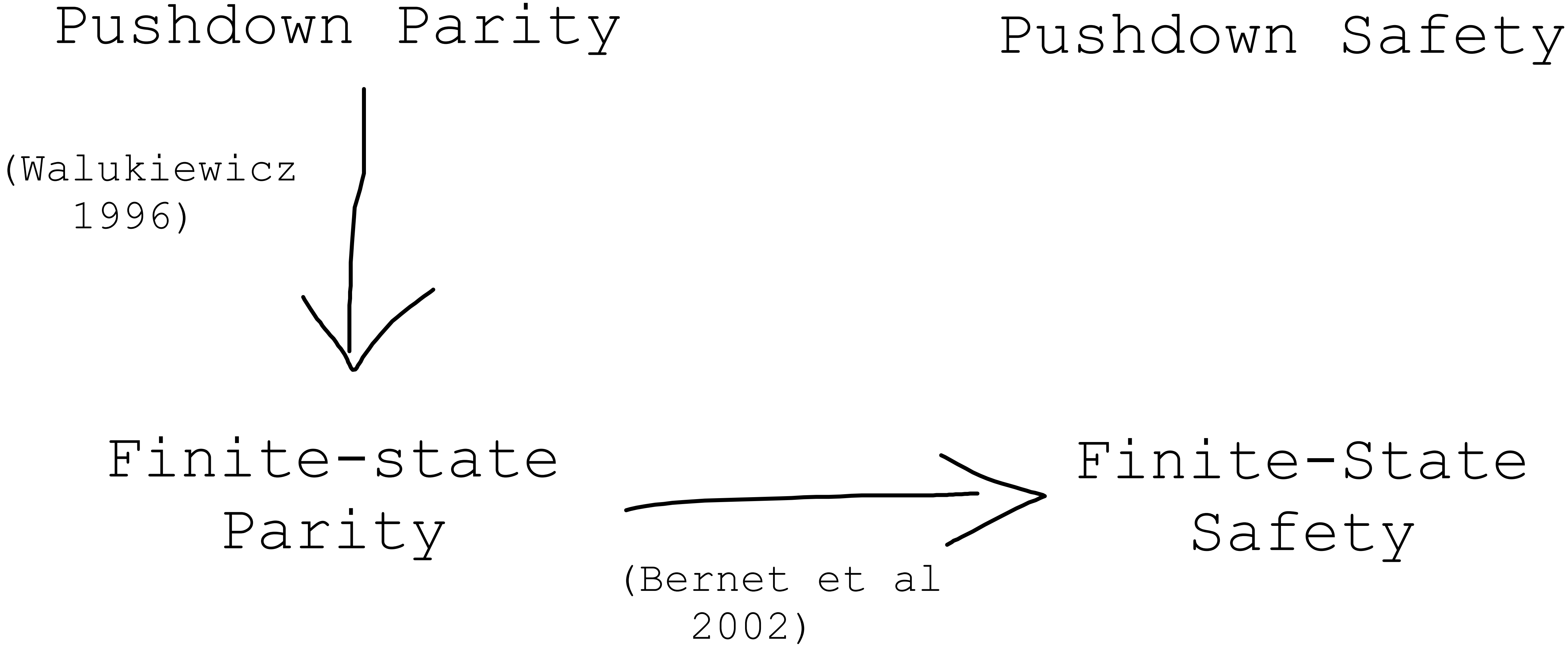
Pushdown Safety

(Walukiewicz 1996)

Finite-state Parity

Finite-State Safety

Pushdown Parity to Safety



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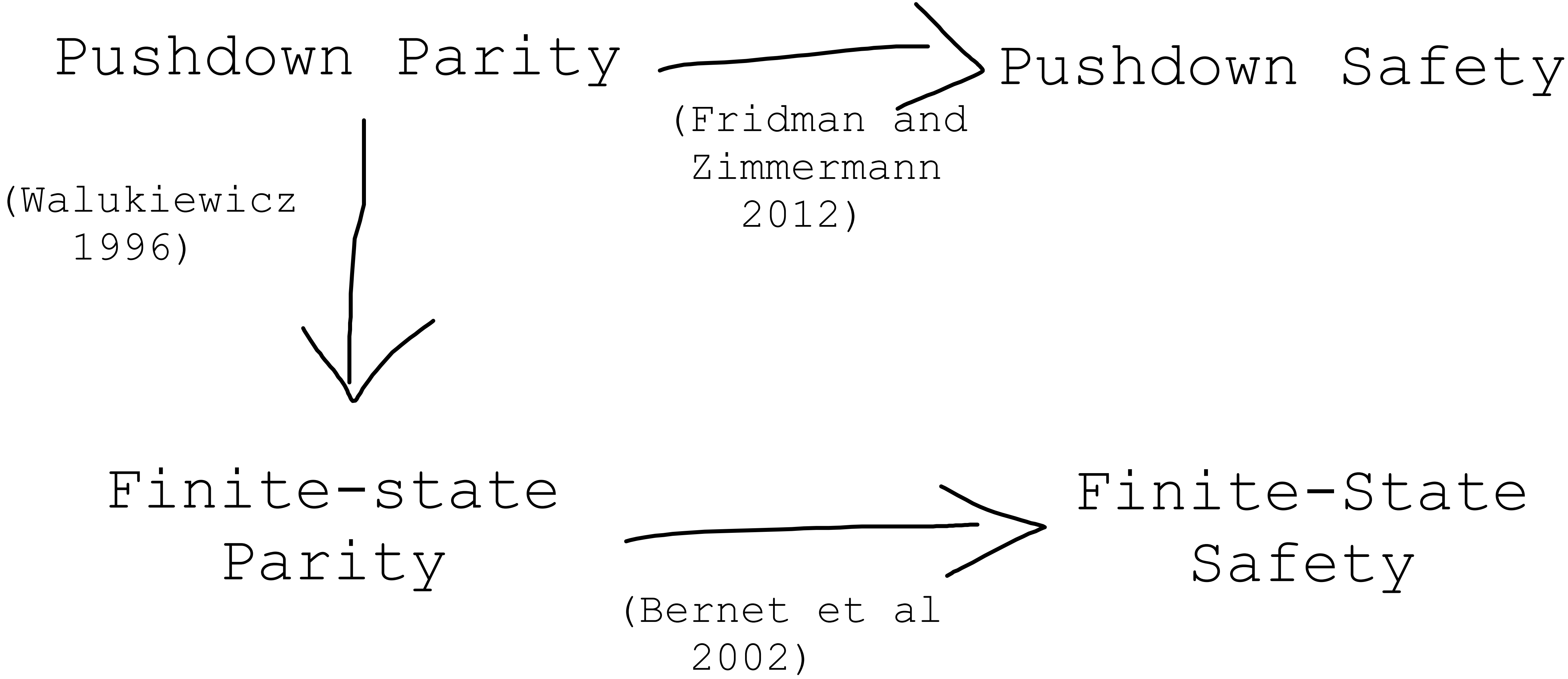

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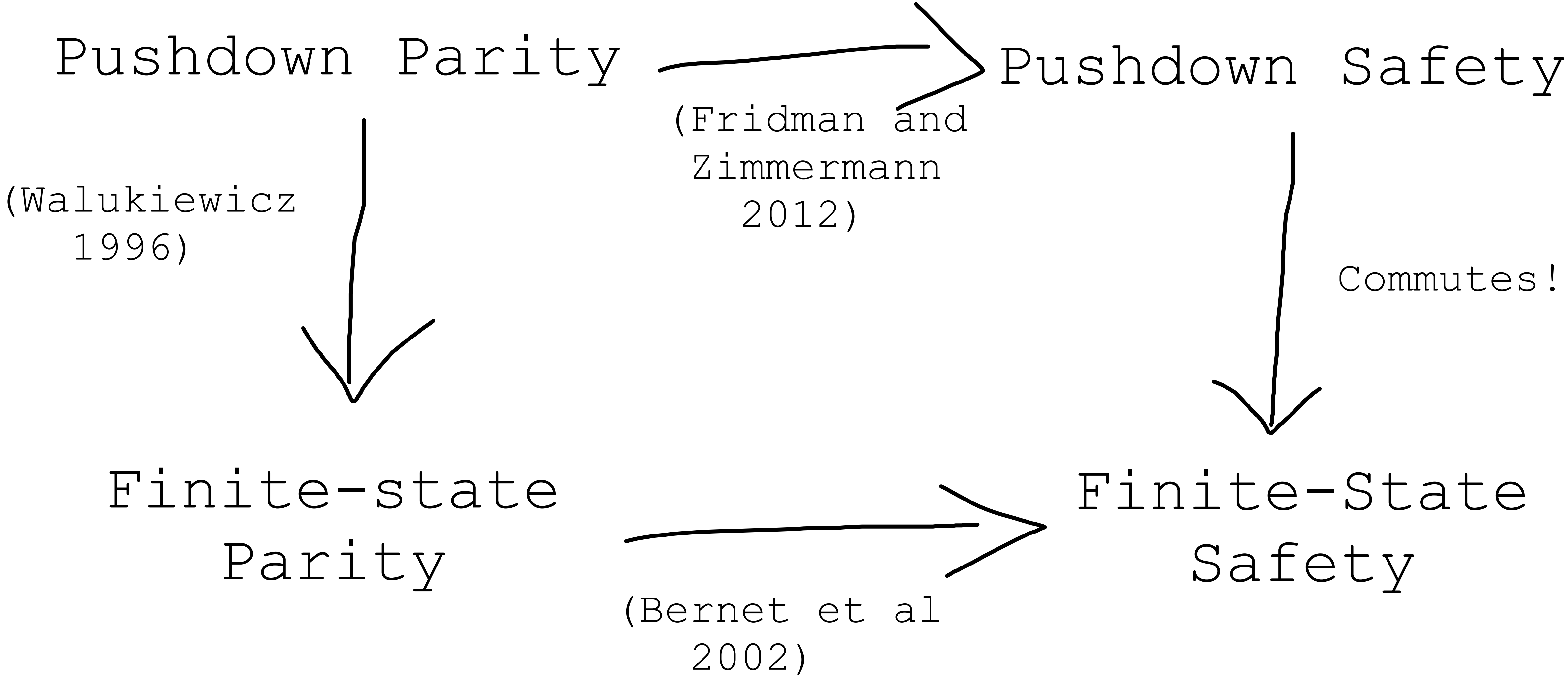
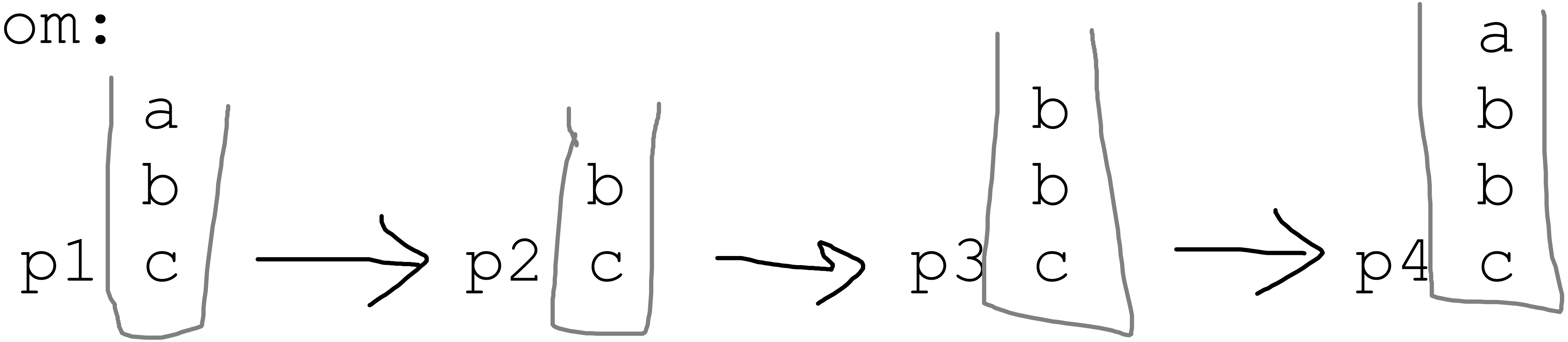


Pushdown Parity to Safety

From:



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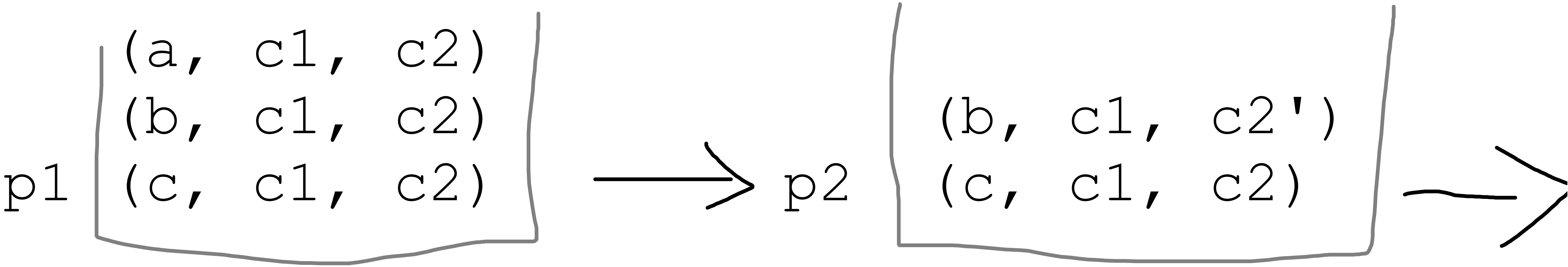

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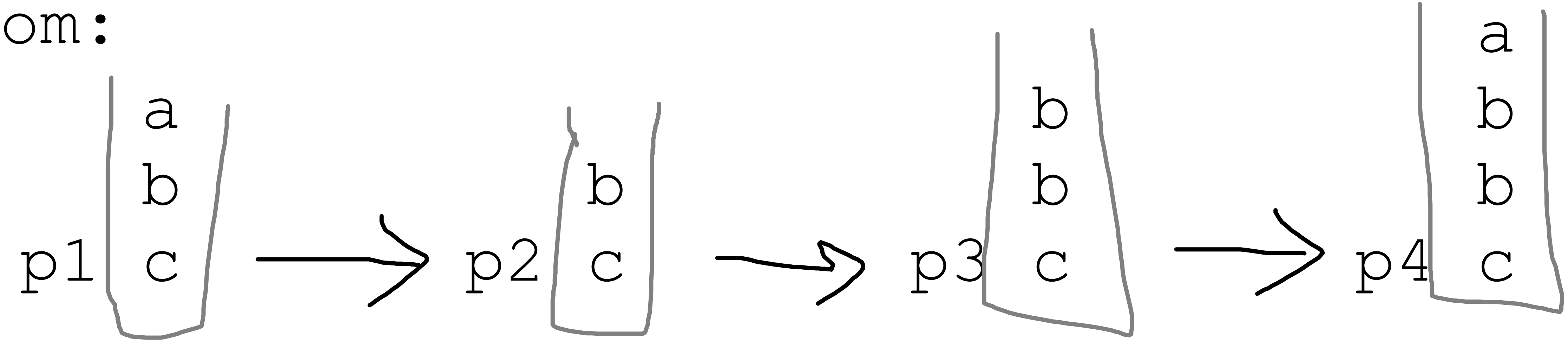

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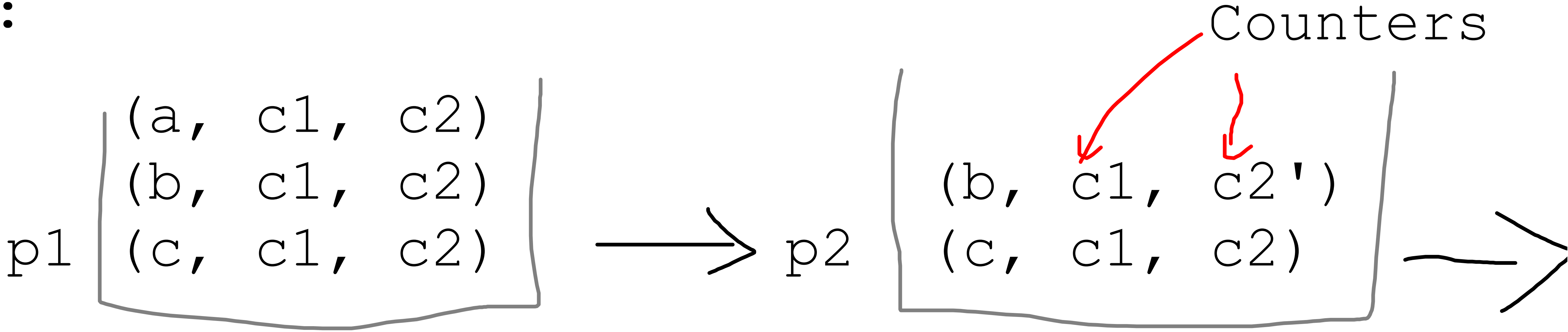

Exponential Blow Up?

```
      (a, c1, c2')
p     (b, c1, c2)
```

- Counter values: can be exponential (Pushdown->finite-state has 2^n states)
- Number of stack characters: (2^n)^k (For k ranks)

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Reducing Size

From:

```
p   (a, c1, c2')
    (b, c1, c2)
```

To:

```
    a
    c2'
    c1
    b
    c2
p   c1
```

Reducing Size

From:

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p   (a, c1, c2')
    (b, c1, c2)
```

To:

```
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    c2'
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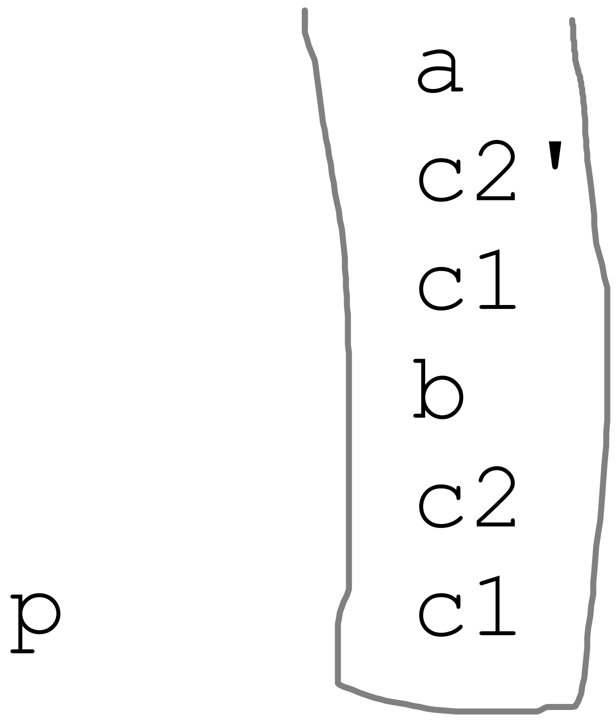
    c2
p   c1
```

Number of characters: 2^n

Updating Counters

Increment c1



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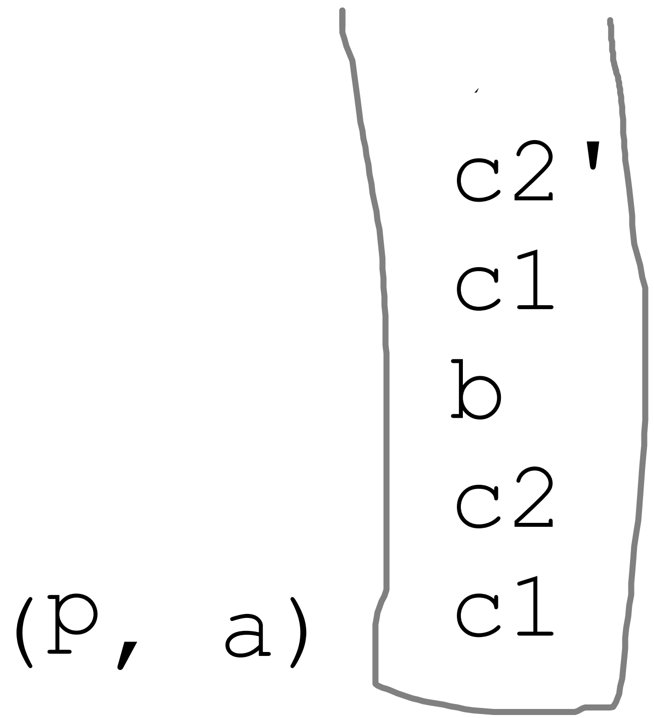

Updating Counters

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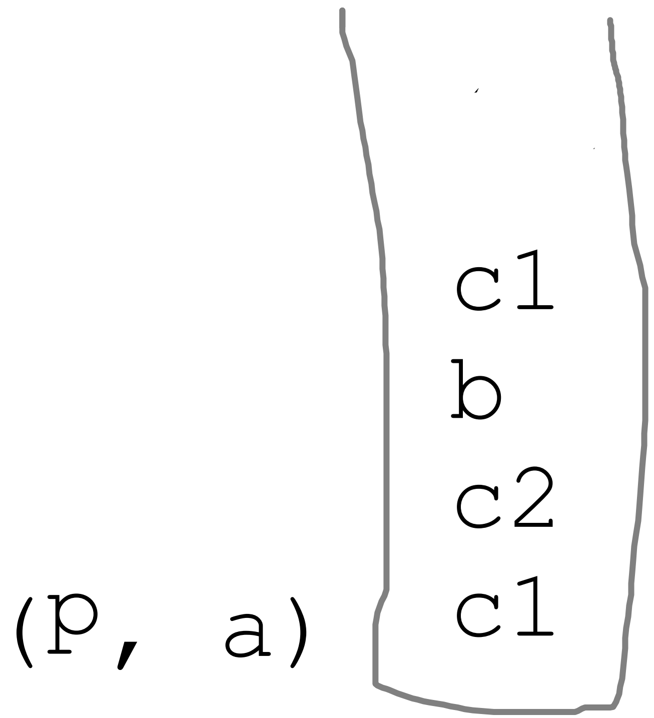

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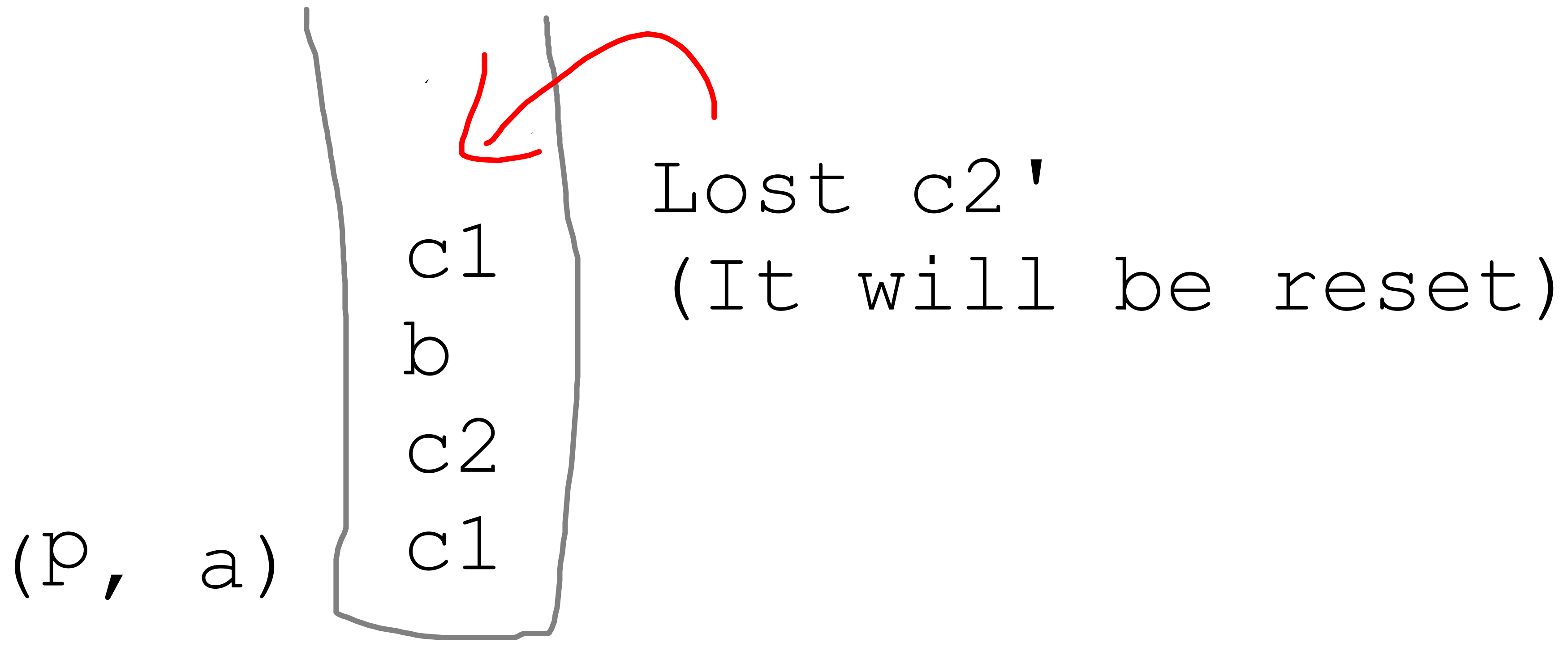

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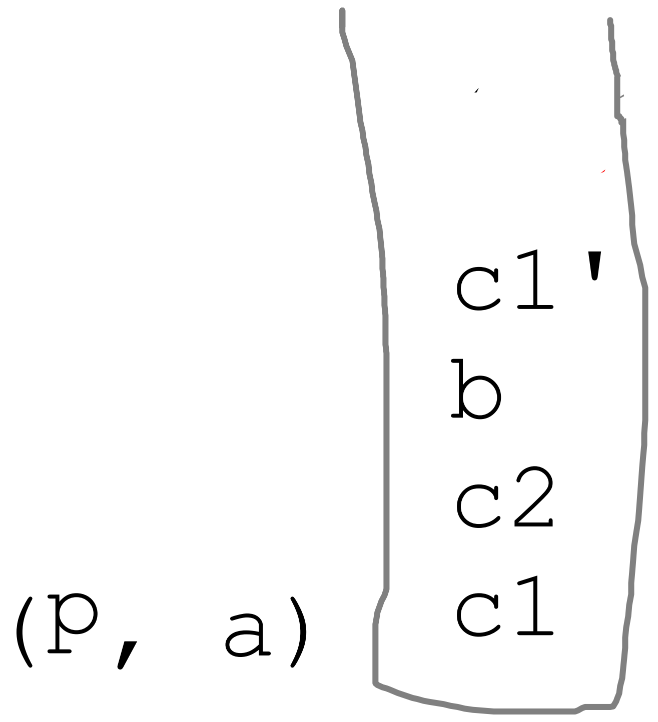

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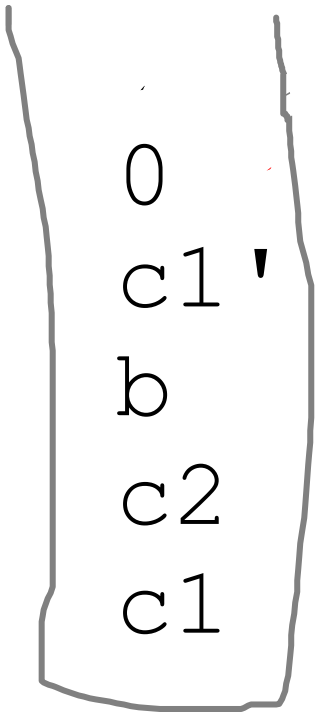

(p, a)

Updating Counters

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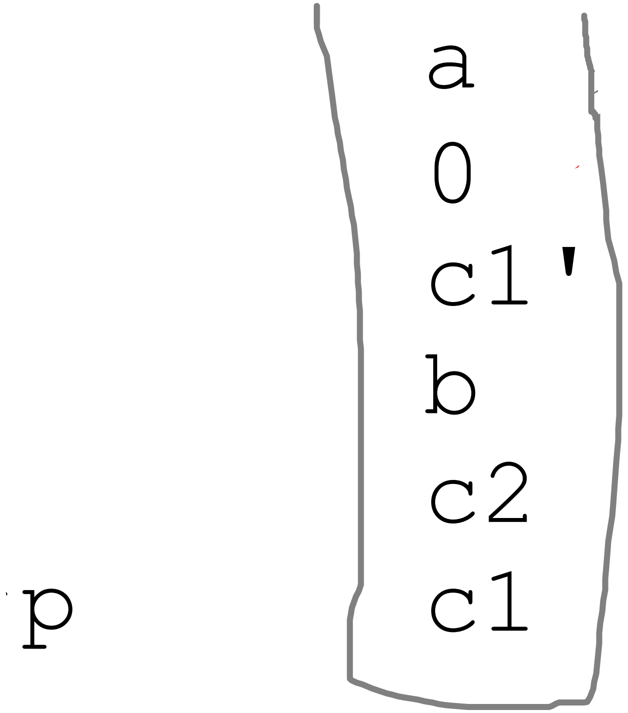
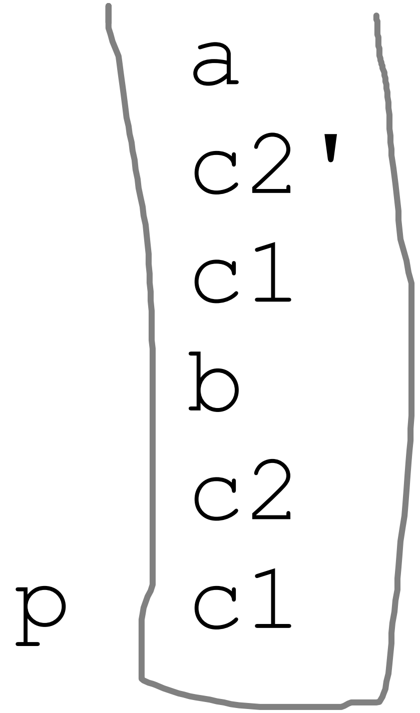


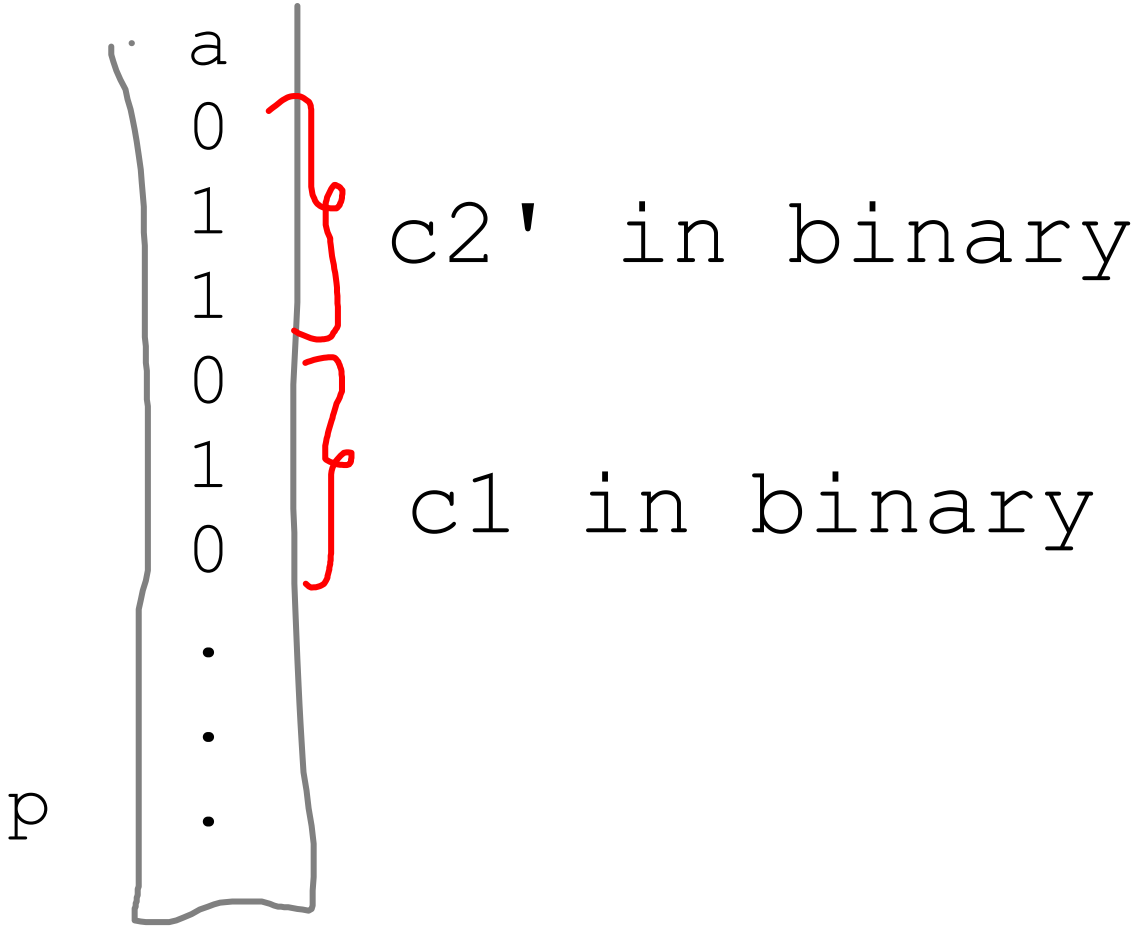
Polynomial No. of Characters

a
c2'
c1
b
c2
p c1

Counters up to 2^n
Alphabet exponential

Polynomial No. of Characters

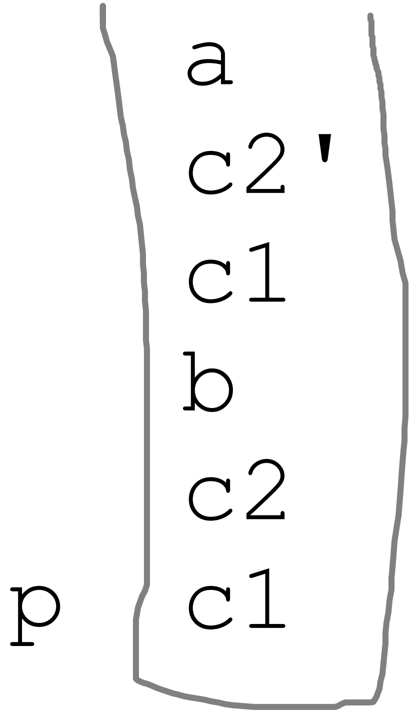




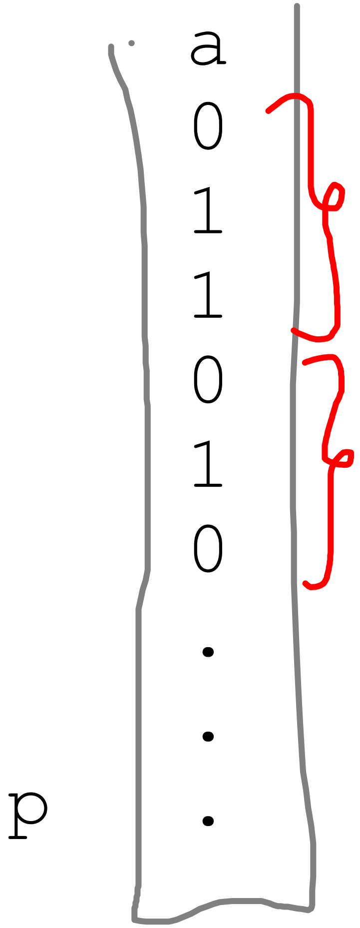
Counters up to 2^n

Alphabet exponential

Polynomial No. of Characters



Counters up to 2^n
Alphabet exponential



c2' in binary

c1 in binary

Pushdowns handle length-n binary

Alphabet polynomial!

Pushdown Parity to Safety

Pushdown parity game -> pushdown safety game

- Naively exponential
- Use stack discipline of counters
- Binary counter encoding of pushdowns
- Polynomial time reduction

Collapsible Pushdown Systems

Pushdown Systems = First-Order Recursion

Collapsible Pushdown Systems
=
Higher-Order Recursion

Higher-Order Programming: Niche?

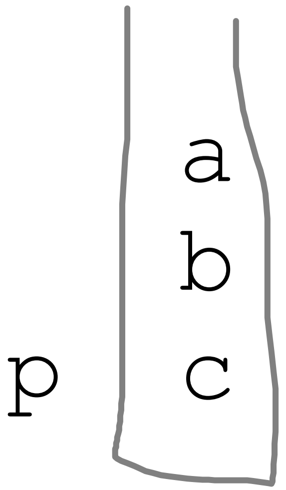
Almost all modern languages support it

- Scala, Go, JavaScript, Python, ...
- Retro-fitted to C++ and Java
- Asynchronous programs/callbacks
- Map/Reduce

Collapsible Pushdown Systems

Higher-order program: functions of functions

Higher-order pushdown: stack of stacks



Collapsible Pushdown Systems

Higher-order program: functions of functions

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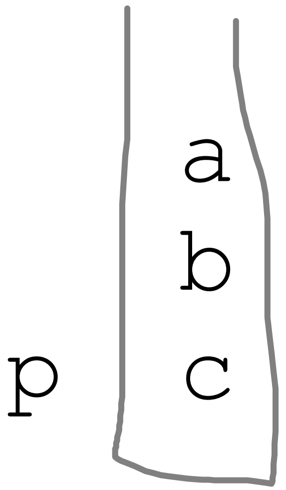
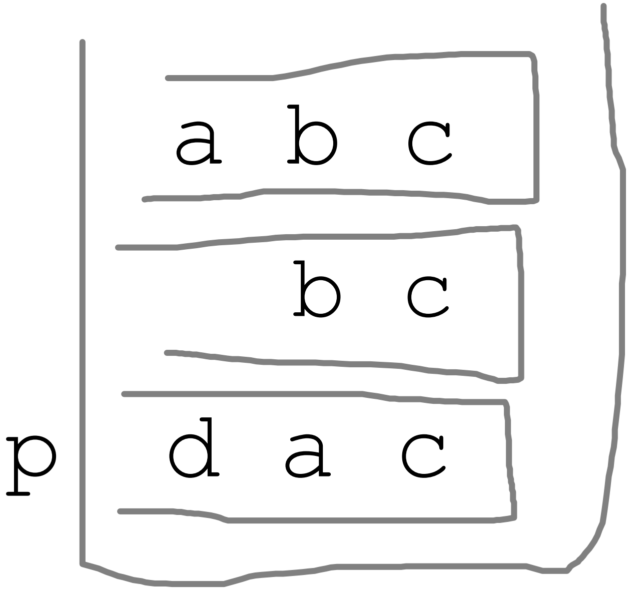
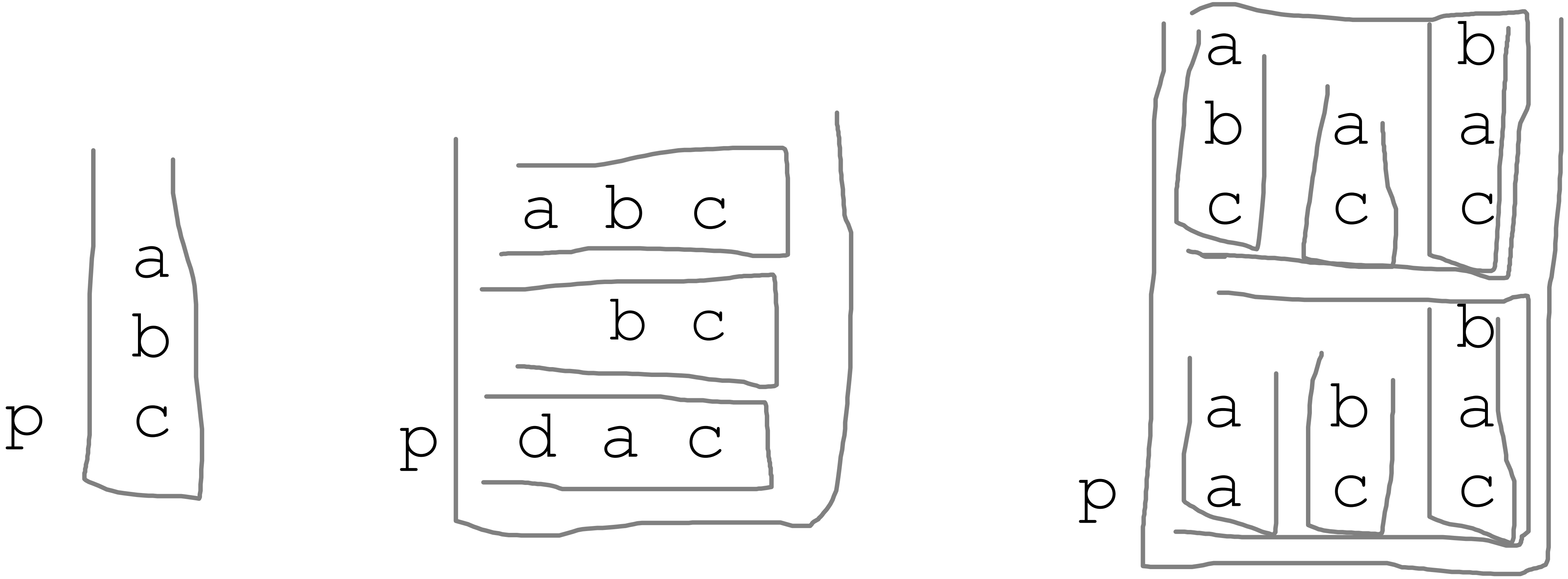




Collapsible Pushdown Systems

Higher-order program: functions of functions

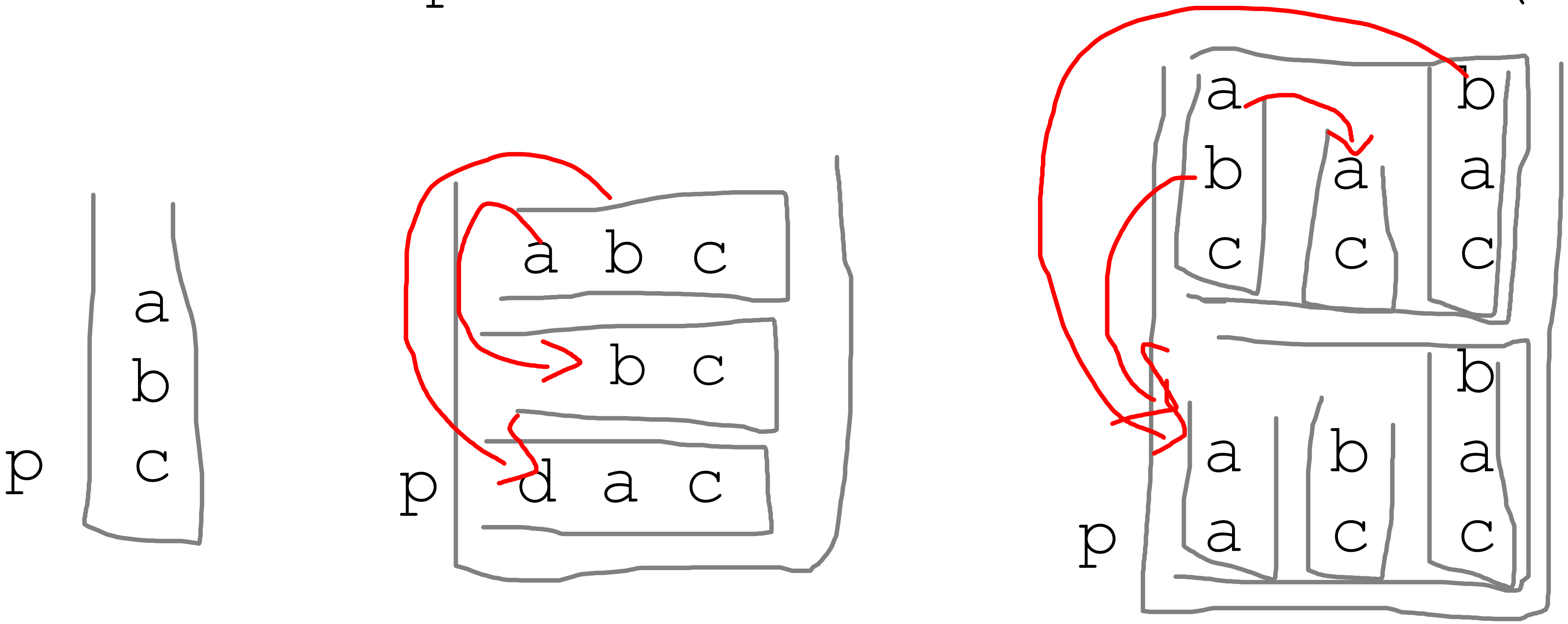
Higher-order pushdown: stack of stacks



Collapsible Pushdown Systems

Higher-order program: functions of functions

Higher-order pushdown: stack of stacks (+links)



Generalising Parity->Safety

- Collapsible Pushdown -> Finite-State (n-Exponential blow up)
- n-Exponential Counters
- Can encode in binary on order-n stack!

Summary

- Parity -> Safety
- Counters look out for loops
  (Even with infinite states)
- Polynomial-time encoding
  - Reduction to finite-state
  - Counters behave like stacks
  - Counter values match limits of system

Future and Related Work

- Can these ideas lead to implementations?
  - Direct implementation
  - Abstraction/refinement of counters
  - Counter size vs. structure of game
  - Backend: HorSat, Preface, &c.
- Parity->Safety:
  - Berwanger and Doyen, 2008
  - Sohail and Somenzi, 2009
  - Biere et al, 2002
  - Podelski and Rybalchenko, 2011
  - Konnov et al, 2017