Exercises to the lecture Logics
Sheet 5

Jun.-Prof. Dr. Roland Meyer Due June 21st, 2013, 12pm

Exercise 5.1 [Semantics of predicate logic]

On the slides, the semantics of a formula $A \in FO(S)$ in a structure $\mathcal{M} = (D, I)$ is defined as a function

$$\mathcal{M}[\![A]\!] : D^V \to \mathbb{B}.$$

Since a function $D^V \to \mathbb{B}$ corresponds to a subset of D^V (and vice versa), we can define the semantics of a formula A as such a subset.

a) Define the semantics of formulae $t_1 = t_2$, $p(t_1, \dots, t_k)$, $\neg A$, $A \wedge B$, $A \vee B$, $\exists x A$, and $\forall x A$, if $\mathcal{M}[\![A]\!]$ is to be a subset of D^V. Hint: Use set operators.

b) Using the new semantics of formulae, how do you define logical consequence $\Sigma \models A$?

Exercise 5.2 [Cardinality of domains]

Given a structure $\mathcal{M} = (D, I)$, we write $|\mathcal{M}|$ for $|D|$, the cardinality of D. We say $\mathcal{M}$ is *finite* if the set D is finite.

a) Present a closed formula A such that $\mathcal{M} \models A$ if and only if $|\mathcal{M}| = 1$.

b) Let B be a formula without the predicate symbol "=". Given a finite interpretation $\mathcal{M}$ with $\mathcal{M} \models B$, how can you construct an interpretation $\mathcal{M}'$ with $|\mathcal{M}'| = |\mathcal{M}| + 1$ and $\mathcal{M}' \models B$? A proof for $\mathcal{M}' \models B$ is not absolutely necessary here.

c) Deduce from b) that there is no formula without "=" that is equivalent to the formula A above.

Exercise 5.3 [A satisfiability check]

a) Present an algorithm that, given a formula A, a finite interpretation $\mathcal{M}$, and a valuation $\sigma \in D^V$, decides whether $\mathcal{M}, \sigma \models A$. *Note:* This means you have shown that satisfaction under a given finite interpretation is decidable.

b) Let A be a closed formula of the form $\exists x_1 \cdots \exists x_n B$, in which B contains no quantifier. Show: If A is satisfiable, then it has a model $\mathcal{M}$ with $|\mathcal{M}| \leqslant |B|$. (We say that the class of formulae of this shape exhibits a *small model property*.)

c) Prove using a) and b): Given a closed formula $A \equiv \exists x_1 \cdots \exists x_n B$ as above, it can be decided algorithmically whether A is satisfiable.

Exercise 5.4 [Modelling]

a) Describe function symbols and predicate symbols that model name, address, and preferred political party of individuals (e.g. in a database). It should be possible that for certain individuals, not all the data is available. In particular, you should specify the arity and the intended meaning of the function and predicate symbols.
b) Formalize the following integrity constraint: "If a person prefers party P or L, then their name and address are available."

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