SS 2013 | 17. April 2013

Exercises to the lecture Logics
Sheet 1

Jun.-Prof. Dr. Roland Meyer | Due 26. April 2013 12:00 Uhr

Exercise 1.1 [Structural Induction]

The *depth* $t(A)$ of a formula A is defined as follows.

- If A is atomic, then $t(A) = 0$.
- If $A \equiv (B * C)$ for a binary connective $*$, then

$$t(A) = \max\{t(B), t(C)\} + 1.$$

- If $A \equiv \neg(B)$, then $t(A) = t(B) + 1$.

Furthermore, let $|A|$ be the length of the formula A, i.e., the number of symbols in A (including parentheses and connectives).

Prove by structural induction that in every correctly bracketed formula

a) the number of opening and the number of closing parentheses coincide.

b) $|A| \leqslant 5k + 1$, where k is the number of occurrences of connectives in A.

c) $|A| \leqslant 4 \cdot 2^{t(A)} - 3$.

Exercise 1.2 [Semantics of formulae]

a) Let φ be a valuation with $\varphi(p) = 1$ and $\varphi(q) = \varphi(r) = 0$. Calculate

$$\varphi(\neg(p \wedge q) \to r)$$

step-by-step using the definition of the evaluation of valuations.

b) Prove or disprove that $q \to (r \to (p \vee q))$ is a tautology.

c) Prove or disprove $q \to p \vDash p \to q$.

d) Prove or disprove $\neg p \vee \neg q \mathrel{\vDash\!\!\!\dashv} \neg(p \wedge q)$.

Exercise 1.3 [Deduction theorem]

a) Let $A_1, \ldots, A_n, B$ be formulae in propositional logic. Show that $A_1 \wedge \cdots \wedge A_n \vDash B$ if and only if $\vDash (A_1 \to (A_2 \to (\cdots (A_{n-1} \to (A_n \to B)) \cdots)))$.

b) Let Σ be a set of formulae and B a formula in propositional logic. Show that $\Sigma \vDash B$ if and only if $\Sigma \cup \{\neg B\}$ is unsatisfiable.

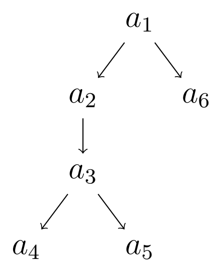
Exercise 1.4 [Paths in rooted trees]

A *rooted tree* is a tree in which one node is chosen as the *root* and the edges are directed such that their source is closer to the root than their target. A *rooted path* is a path that starts in the root (but does not necessarily end in a leaf). For each rooted path P, we write $\hat{P}$ for the set of nodes it meets. A subset of nodes is called *rooted path set* if it is of the form $\hat{P}$ for some rooted path P.

Let $V = \{a_1, \ldots, a_n\}$ be the nodes of a rooted tree and let $p_1, \ldots, p_n$ be atomic formulae. The subsets of V and the valuations on $p_1, \ldots, p_n$ are in one-to-one correspondence, where the set $S \subseteq V$ corresponds to the valuation φ for which

$$\varphi(p_i) = 1 \text{ if and only if } a_i \in S$$

for each $i \in \{1, \ldots, n\}$.



a) For the rooted tree on the right, present a formula A for which $\varphi(A) = 1$ if and only if φ corresponds to a rooted path set.

b) Devise a general method that, given a rooted tree T, constructs a formula A such that $\varphi(A) = 1$ if and only if φ corresponds to a rooted path set in T.

Delivery: until 26. April 2013 12:00 Uhr into the box next to room 34/401.4