Games with perfect information
Exercise sheet 11

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Summer term 2018

Out: June 27 Due: July 4

Submit your solutions on Wednesday, July 4, at the beginning of the lecture.
Please submit in groups of three persons.

Exercise 1: Counter machines

Show how to construct a counter machine of dimension $d \geqslant 2$ with two control states q_0, q_f such that there is a transition sequence from $(q_0, n, m, \ldots)$ that reaches q_f if and only if

a) $n \geqslant m$,

b) $n < m$,

c) n is divisible by m .

Hint: You may use an arbitrary constant number of additional counters.

Exercise 2: Primality testing

Show how to construct a counter machine of dimension $d \geqslant 1$ with two states q_0, q_f such that **there is** a transition sequence from $(q_0, n, \ldots)$ that reaches state q_f if and only if n is **not** a prime number. Explain your construction.

Hints: You may use an arbitrary constant number of additional counters. You can use non-determinism. You may split your construction into smaller parts (*gadgets*) and explain later how these should be combined.

Exercise 3: One-counter automata as pushdowns
Prove that one-counter automata can be simulated by pushdown systems.

Recall that a pushdown system is an automaton $(Q, \to)$ with memory S^*, where S is some finite **stack alphabet**. The transition rules in $\to$ are of the shape

$$q \xrightarrow{\text{push } a} p \quad \text{or} \quad q \xrightarrow{\text{push } a} p$$

for symbols $a \in S$. There is a transition $((q, m) \to (p, m')) \in T$ if

- there is a rule $q \xrightarrow{\text{push } a} p$ and $m' = m.a$, or
- there is a rule $q \xrightarrow{\text{pop } a} p$ and $m = m'.a$.

(Here, we use the convention that the right end of the word m encodes the top of stack.) Note that a pop a transition is only enabled when a is indeed the top of stack.

Assume that some one-counter automaton $A_{\text{OCA}} = (Q', \to')$ with states q_0, q_f is given. Show how to construct a pushdown system $A_{\text{PDS}} = (Q, \to)$ with $Q' \subseteq Q$ over a suitable stack alphabet such that q_f is reachable in A_{OCA} from $(q_0, 0)$ if and only if q_f is reachable in A_{PDS} from some suitable initial configuration. Briefly argue that your construction is correct.

Exercise 4: Integer counter machines
An **integer counter machine** of dimension d is defined similarly to a counter machine of dimension d. However, the counters can reach negative values, i.e. the memory is $\mathbb{Z}^d$. A transition of type $q \xrightarrow{x_i--} p$ is enabled even if the value of counter x_i is zero.

a) Let A_{ICM} be an integer counter machine of dimension d, and q_0, q_f control states. Show how to construct a counter machine A_{CM} with states q_0' and q_f' such that q_f is reachable from $(q_0, 0, \ldots, 0)$ in A_{ICM} if and only if q_f' is reachable from $(q_0', 0, \ldots, 0)$ in A_{CM}.

b) Let A_{CM} be a counter machine of dimension d, and q_0', q_f' control states. Show how to construct an integer counter machine A_{ICM} with states q_0 and q_f such that q_f is reachable from $(q_0, 0, \ldots, 0)$ in A_{ICM} if and only if q_f' is reachable from $(q_0', 0, \ldots, 0)$ in A_{CM}.

In both cases, argue briefly that your construction is correct.