Games with perfect information
Exercise sheet 8

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Out: May 30 Due: June 6

Submit your solutions for the Exercises 2 - 4 on Wednesday, June 6, at the beginning of the lecture. Please submit in groups of three persons.

Exercise 1: Read it up! (no submission)

Please read and understand:

a) The explanation on A_{branches} and B in the proof of Rabin's tree theorem (pages 89–90 of the lecture notes).

b) The definition of S2S, monadic second-order logic with 2 successors (pages 106–109 of the lecture notes).

Exercise 2: Complementing a PTA

In this exercise, you should apply Rabin's tree theorem to the automaton A_1 from Part a) of Exercise 8.42 (Exercise 1 form Exercise sheet 7).

Recall its definition: $A_1 = (\Sigma, \{q_0, q_1\}, q_0, \rightarrow, \Omega)$ with

$$\begin{aligned} \rightarrow_a &= \{(q_0, (q_1, q_1))\}\ , \\ \rightarrow_b &= \{(q_1, (q_0, q_0))\}\ , \\ \Omega(q_0) = \Omega(q_1) &= 0\ . \end{aligned}$$

a) Construct the set $S = Q^{\leq n} \rightarrow D$.

Hint: To avoid the following construction becoming excessively large, restrict the domain to vectors of states that can actually occur.

b) Construct the parity word automaton $A_{branches}$.

c) Make $A_{branches}$ deterministic by adding an error-state and the corresponding transitions. (For each symbol a, s, d, and each state q, there needs to be exactly one transition $(q, q') \in\rightarrow_{a,s,d}$.) Complement $A_{branches}$ to obtain the automaton B with $\mathcal{L}(B) = \overline{\mathcal{L}(A_{branches})}$.

d) Construct the parity tree automaton C for $\mathcal{L}'$ that simulates B on all branches of a tree.

e) Project C to Σ to obtain the automaton $\overline{A_1}$. Check that $\mathcal{L}\left(\overline{A_1}\right) = \overline{\mathcal{L}(A_1)}$ indeed holds by describing the language of $\overline{A_1}$.

Exercise 3: Understanding S2S formulas

For each of the following S2S formulas, characterize the set of trees that satisfy them. Argue that your characterization is correct.

a)

$$\exists x\colon P_a(x) \wedge \forall y\colon x \neq y \rightarrow P_b(y) \ .$$

b)

$$P_a(\varepsilon) \wedge \forall x \forall y \forall z\colon (S_0(x,y) \wedge S_1(x,z)) \rightarrow ((P_a(x) \rightarrow P_b(y) \wedge P_b(z)) \wedge (P_b(x) \rightarrow P_a(y) \wedge P_a(z))) \ .$$

Exercise 4: Writing S2S formulas

Consider the Alphabet $\Sigma = \{a_{/2}, b_{/2}\}$. Our goal is to create a closed S2S-formula for the language $\mathcal{L}$ of trees in which exactly one branch contains infinitely many as (known from Part c) of Exercise 8.42 / Exercise 1 from Exercise sheet 7).

a) Consider the following S2S formula that has the free second-order variable X.

$$\begin{aligned}
\text{Branch}(X) \quad = \quad & \varepsilon \in X & (1)\\
\wedge \quad & \forall x\colon x \in X \rightarrow \exists y \exists z\colon S_0(x,y) \wedge S_1(x,z) \wedge (y \in X \oplus z \in X) & (2)\\
\wedge \quad & \forall y\colon (y \in X \wedge y \neq \varepsilon) \rightarrow \exists x\colon x \in X \wedge (S_0(x,y) \vee S_1(x,y)) & (3)
\end{aligned}$$

Here, $\oplus$ is XOR and $\rightarrow$ is implication. They can be easily rewritten using negation, conjunction, and disjunction.

Argue that Branch(X) evaluates to true under a structure $\mathcal{S}(\mathcal{T})$ and an interpretation $\mathcal{I}_\mathcal{T}$ if and only if $\mathcal{I}_\mathcal{T}(X)$ is a set of positions that forms a branch of $\mathcal{T}$. Explain the purpose of each Line (1) - (3).

b) In S2S, we only have an equality predicate for first-order terms. Construct a formula Equal(X, Y) with two free second-order variables X, Y that evaluates to true under a structure $\mathcal{S}(\mathcal{T})$ and an interpretation $\mathcal{I}_\mathcal{T}$ if and only if $\mathcal{I}_\mathcal{T}(X) = \mathcal{I}_\mathcal{T}(Y)$.

c) Construct formulas $\text{Fin}_a(X)$ respectively $\text{Inf}_a(X)$ with one free second-order variable X that evaluate to true under a structure $\mathcal{S}(\mathcal{T})$ and an interpretation $\mathcal{I}_\mathcal{T}$ if and only if $\mathcal{I}_\mathcal{T}(X)$ contains only finitely many respectively infinite many nodes labeled by a.

For simplicity, you may suppose that $\mathcal{I}_\mathcal{T}(X)$ is a branch of $\mathcal{T}$.

d) Combine the previous parts of this exercises to construct a closed S2S-formula $\varphi_\mathcal{L}$ that evaluates to true under a structure $\mathcal{S}(\mathcal{T})$ if and only if $\mathcal{T} \in \mathcal{L}$.