Games with perfect information
Exercise sheet 3

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Out: April 18 Due: April 25

Submit your solutions on Wednesday, April 25, during the lecture.
Please submit in groups of three persons.

Exercise 1: 2×2 **tic tac toe**

Consider a 2×2-variant of tic tac toe, i.e. tic tac toe played on a 2×2 matrix. We assume that $\bigcirc$ starts. The player that is first able to put 2 of her marks into one row, column or diagonal wins, and the game then stops.

Formalize this game as a reachability game, draw the game arena as a graph, and solve it explicitly using the attractor algorithm.

Exercise 2: Determinacy of games of finite length

Let $\mathcal{G} = (G, win)$ be a game such that each maximal play of $\mathcal{G}$ has finite length. Prove that $\mathcal{G}$ is determined, i.e. every position is winning for exactly one of the players, $V = W_{\bigcirc} \uplus W_{\square}$.

Hint: Construct a reachability game whose set of positions is $Plays^{\mathcal{G}}$.

Exercise 3: Attractors have attractive algorithmics

a) Prove that if $\mathrm{Attr}^{i}_{\star}(B) = \mathrm{Attr}^{i+1}_{\star}(B)$, then we have $\mathrm{Attr}^{i}_{\star}(B) = \mathrm{Attr}_{\star}(B)$.

   Conclude that if the set of positions V is finite, we have $\mathrm{Attr}_{\star}(B) = \mathrm{Attr}^{|V|}_{\star}(B)$.

b) Let $G = (V, E)$ be a finite game arena, and let $B \subseteq V$ be a set. We consider the reachability game on G with respect to B.

   Write down pseudo-code for an algorithm that computes the winning region $W_{\bigcirc}$ of the existential player, and at the same time computes uniform positional winning strategies $s_{\bigcirc}, s_{\square}$ for both players.

Exercise 4: Graphs with infinite out-degree

In the section on reachability games, we made the assumption that the out-degree of the game arena is finite. In this exercise, we want to understand this restriction.

Let $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ denote the positive natural numbers. We consider the infinite graph $G = (V, R)$ given by

$$V = \{start, goal\} \uplus \bigcup_{i \in \mathbb{N}^+} Path_i\ , \text{ where for each } i \in \mathbb{N}^+, \text{ we have } Path_i = \left\{p^i_1, p^i_2, \ldots, p^i_i\right\},$$

$$R = \bigcup_{i \in \mathbb{N}^+} \left\{\left(start, p^i_1\right)\right\} \uplus \bigcup_{i \in \mathbb{N}^+} \left\{\left(p^i_i, goal\right)\right\} \uplus \bigcup_{i \in \mathbb{N}^+} \bigcup_{j=1}^{i-1} \left\{\left(p^i_j, p^i_{j+1}\right)\right\}.$$

We want to consider a reachability game on G with respect to the winning set $\{goal\}$, i.e. $\bigcirc$ needs to reach the position $goal$, $\square$ wants to prevent this.

a) Draw a schematic representation of the graph G, e.g. involving the vertices $\{start, goal\}$ and the positions in $Path_i$ for $i \leqslant 4$.

b) Assume that all positions are owned by the existential player. For each position $x \in V$, give the minimal $i_x \in \mathbb{N}$ such that $x \in \mathrm{Attr}^{i_x}_{\bigcirc}(\{goal\})$, respectively $i_x = \infty$ if no such i_x exists.

   Present a winning strategy for the reachability game from the position $start$.

c) Assume that all positions are owned by the universal player. For each position $x \in V$, give the minimal i_x such that $x \in \mathrm{Attr}^{i_x}_{\bigcirc}(\{goal\})$, respectively $i_x = \infty$ if no such i_x exists.

   Which player wins the reachability game from $start$?