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Games with perfect information
Exercise sheet 13

TU Braunschweig
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Out: July 7 Due: July 14

Submit your solutions until Friday, July 14, 14:00, in the box next to office 343.

As there will be no exercise class for this sheet, we will make solutions available.

This is a **bonus exercise sheet**: Its points will count towards your number of achieved points, but not towards the maximal number of points.

Exercise 1: Gale-Stewart games as graph games

Let $\mathcal{G}(A, B)$ be a Gale-Stewart game. Define an equivalent game over a graph with the set of positions

a) $V = A \times \{\square, \bigcirc\}$,

b) $V = A^*$.

In each case, specify the ownership, the arcs, the winning condition, and the initial position of interest.

Remark 2: Notation for (sets of) sequences

We recall the notations needed for the next two exercises.

Let V be a set. We denote by V^* the set of sequences over V of finite length, by V^ω the set of sequences over V^ω of infinite length.

Let $p', p'' \in V^*, p \in V^\omega$. Finite sequences p', p'' can be concatenated, resulting in the finite-length sequence $p'.p''$. A finite sequence p' can be concatenated with the infinite sequence p, resulting in the infinite sequence $p'.p$.

For sets of sequences, we define their concatenation element-wise.
Let $K', K'' \subseteq V^*$ and $H \subseteq V^\omega$. We define

$$K'.K'' = \{p'.p'' \in V^* \mid p' \in K', p'' \in K''\} ,$$
$$K'.H = \{p'.p \in V^\omega \mid p' \in K', p \in H\} .$$

We identify elements $x \in V$ with the sequence $x \in V^*$ of length one.

For a sequence $p' \in V^*$, we write p' to denote the singleton set $\{p'\} \subseteq V^*$.

Exercise 3: Reachability games as Gale-Stewart games

Let $\mathcal{G}$ be a reachablity game, specified as usual by a game arena $G = (V_\Box \uplus V_\bigcirc, R)$ and a winning set $V_{reach} \subseteq V$. For simplicity, let us assume that G is bipartite and the player take turns alternately. Furthermore, we fix the initial position $x_0 \in V_\bigcirc$.

Our goal is to create an equivalent Gale-Stewart game $\mathcal{G}\big(V, B\big)$, where B is of the shape

$$B = (B_\bigcirc \cup B_{\neg x_0}) \setminus (B_\Box \cup B_{reach}) .$$

a) We define

$$B_\bigcirc = \bigcup_{\substack{x \in V_\Box, \\ y \in V_\bigcirc, \\ (x,y) \notin R}} \left\{ p \in V^\omega \,\middle|\, p \in V^{odd}.x.y.V^\omega \right\} .$$

Here, $V^{odd} \subseteq V^*$ should denote the set of all finite sequences over V of odd length.

Argue that an infinite play $p \in V^\omega$ of $\mathcal{G}\big(V, B\big)$ is in $B_\bigcirc$ if and only if refuter makes a move that is illegal, i.e. not corresponding to an arc in the graph. This means that there is a prefix of the play of the shape $p' = p''.x$ in which refuter y such that $(x, y) \notin R$.

b) Define the set $B_{\neg x_0}$ of plays that are not starting in x_0.

c) Define the set $B_\Box$ of all plays in which prover makes an illegal move.

d) Define the set B_{reach} of all plays in which at least one position in the set V_{reach} occurs.

Exercise 4: Open sets

Let A be a set. We call a set $B \subseteq A^\omega$ of infinite sequences over A **open** if it is of the shape

$$B = K.A^\omega$$

for some set $K \subseteq A^*$ of finite sequences over A. (This essentially means that a set B is open if the membership of a play $p \in A^\omega$ in B is determined by a finite prefix of B.)

a) Prove that the empty set $\varnothing \subseteq A^\omega$ and A^ω itself are open.

b) Prove that if B and B' are open, then also their union $B \cup B'$ is open.

c) Prove that if B and B' are open, then also their intersection $B \cap B'$ is open.

Remark: The parts a) - c) almost prove that the notion of being open as defined here defines a topology on A^ω. (It would remain to prove that arbitrary unions of open sets are open, which can also be done.) In fact, it defines the so-called **product topology** on A^ω, where we use the **discrete topology** on A (meaning each subset of A is defined to be open).

Remark: The sets $B_\bigcirc, B_{\neg x_0}, B_\Box, B_{reach}$ from the previous exercise are open.