# 3. VASS Reachability

**Goal:** Introduce the decision procedure for VASS reachability.

## 3.1 Marked Graph Transition Sequences

*Recall:*

The decision procedure for VASS reachability

is an abstraction refinement algorithm:

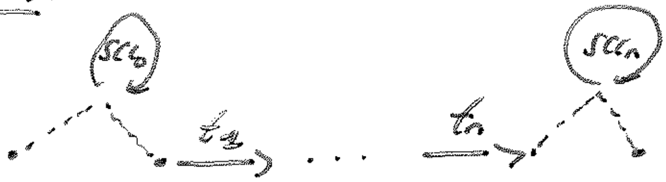
$$L_{\mathbb{Z}}(S_0) \supseteq L_{\mathbb{Z}}(S_1) \supseteq \dots \supseteq L_{\mathbb{N}}(V)$$

*Goal:*

Introduce the VASS variant

that occurs in the sets $S_i$.

*Idea:*




*Definition:*

- A precovering graph $G = (V, (q_r, c_{in}), (q_r, c_{out}), e)$

  consists of • a strongly connected VASS $V = (Q, \Sigma, C, T)$

  with • initial configuration $(q_r, c_{in})$ — root of the precovering graph

  • final configuration $(q_r, c_{out})$

  and $e: Q \to \mathbb{N}_\omega^C$ a consistent assignment

  of generalized markings to nodes.

- Consistency means

  (i) all nodes agree on the counters

  that should have $\mathbb{N}$-values:

there is a subset of counters $D \subseteq C$
so that for all $q \in Q$. $\mathcal{C}(q)[d] \in \mathbb{N}$ iff $d \in D$.

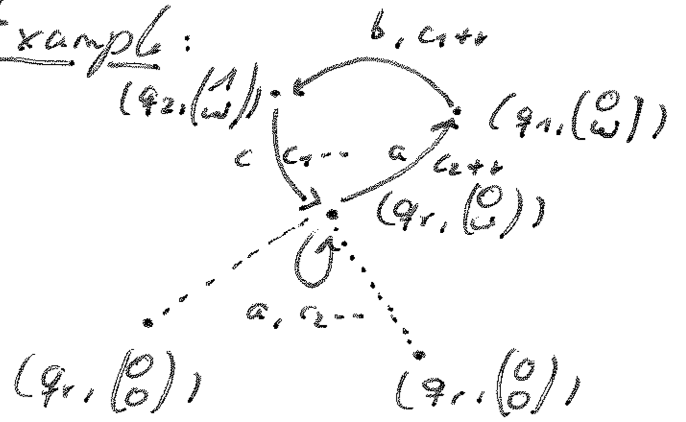
(ii) the consistent assignment tracks the effect of transitions:
for all $\underbrace{(q_1, a, y, q_2)}_{t} \in T$,
we have
$$\mathcal{C}(q_2) = \mathcal{C}(q_1) + \overbrace{y}^{eff(t)}.$$

(iii) every counter that carries a concrete value in the precovering graph, has the concrete value in the root as the initial and final value:
$$\mathcal{C}(q_r)[D] = c_{in}[D] = c_{out}[D].$$

• We use $\Omega(G) = C \setminus D$
for the counters that are decorated $\omega$ in the precovering graph.
We use $\Omega_{in} = \Omega(G) \setminus \Omega(c_{in})$
$\Omega_{out} = \Omega(G) \setminus \Omega(c_{out})$
for the counters that are $\omega$ in $G$, but not in the initial resp. final configuration.

<u>Illustration:</u>
In the above picture,  denotes a precovering graph.

Example:




$C = \{c_1, c_2\}$

$\Omega = \{c_1\}$

$\Omega(G) = \{c_2\}$

$\Omega_{in} = \{c_2\}$

$\Omega_{out} = \{c_2\}$

$$c_{in}[\Omega] = 0 = c_{out}[\Omega] = \underset{\binom{0}{\omega}}{\underset{\shortparallel}{C(q_r)}}[\Omega]$$

## ω-Values:

- A counter may be $\omega$ in a precovering graph, but have a concrete initial value.
- Then we should be able to pump this value while going from the root back to the root.
- Pumping means $CSup(G) \neq \emptyset$ with

$$CSup(G) = \{\sigma \in T^* \mid \exists c_1, c_2 \in \mathbb{N}^C.\ c_1 \leq_\omega c_{in} \wedge c_1 <^{\Omega_{in}} c_2 \wedge (q_r, c_1).\sigma.(q_r, c_2) \in Runs_{\mathbb{N}}(G)\}.$$

- Here, $c_1 \leq_\omega c_{in}$ is the <u>specialization preorder</u> with

$$\forall c \in C.\quad c_1[c] = c_{in}[c] \in \mathbb{N} \quad \text{// concrete values match}$$
$$\vee\ c_{in}[c] = \omega \quad \text{// } \omega\text{-values may be concretized.}$$

Moreover, $c_1 <^{\Omega_{in}} c_2$ says $c_1[c] < c_2[c]$ f.a. $c \in \Omega_{in}$.

Note that we also have $c_1 \leq c_2$ as we require to go through a loop rooted in $q_r$.

Example

$$\overset{\binom{0}{0} \geq_\omega}{(q_r, \underset{c_1}{\binom{0}{0}})} \xrightarrow[c_2++]{a} (q_1, \binom{0}{1}) \xrightarrow[c_1++]{b} (q_2, \binom{1}{1}) \xrightarrow[c_1--]{c} (q_r, \underset{c_2}{\binom{0}{1}})$$

$$c_1 <^{\Omega_{in}} c_2$$

· We also need to be able to pump down:

$$CS_{down}(G) = CS_{up}(G^{rev})^{rev} \neq \emptyset.$$

The reverse of an edge is

$$(q_1, a, x, q_2)^{rev} = (q_2, a, -x, q_1),$$

so we decrement where we have incremented and vice versa.

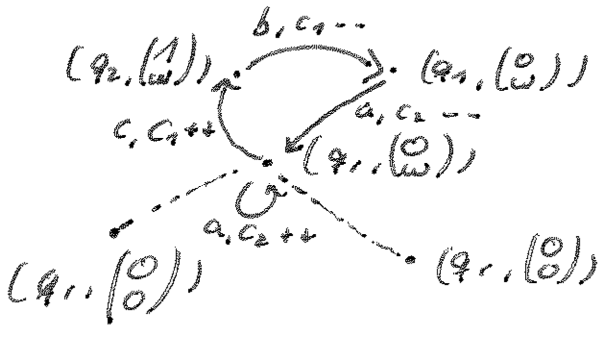
$$G^{rev} = (V, (q_r, c_{in}), (q_r, c_{out}), \mathcal{E})^{rev}$$

$$= (\underbrace{V^{rev}}_{\text{reverse all edges}}, (q_r, c_{out}), (q_r, c_{in}), \mathcal{E}).$$

Note that $S_{out}$ becomes $S_{in}$ in $G^{rev}$.

<u>Example:</u>

$G^{rev}$




We have $(q_r, \binom{0}{0}) \xrightarrow[c_2++]{a} (q_r, \binom{0}{1}) \in CS_{up}(G^{rev})$,

which means $(q_r, a, c_2--, q_r) \in CS_{down}(G)$.

<u>Lemma:</u>

$CS_{up}(G) \neq \emptyset$ can be checked as unboundedness.
down

<u>Proof:</u> Build the coverability graph and find the required omegas. Note that for VASS, the control state has to match upon repetitions. □

Definition:

- The set of <u>marked graph transition sequences</u> (MGTS) is defined by

$$W ::= G \mid W_1 . \underbrace{(\overset{\Sigma}{a}, \overset{\mathbb{Z}^C}{x})}_{up} . W_2 .$$

In an MGTS, all precovering graphs share the same alphabet $\Sigma$ and the same set of counters $C$.

The states of the precovering graphs are pairwise disjoint. (have initial and final configurations)

- We also understand MGTS as (initialized) VASS:

$W.Q$ = nodes $\qquad$ $W.cin = W[first].cin$

$W.\Sigma$ = alphabet $\qquad$ $W.cout = W[last].cout$.

$W.C$ = counters

$W.T$ = transitions

- MGTS have their own notion of <u>intermediate acceptance</u>, where the values at entry and exit nodes of precovering graphs have to be reached up to $\leq \omega$.

Definition:

$IACC_{\mathbb{Z}}(W)$ is the set of all runs $\rho \in Runs_{\mathbb{Z}}(W)$, so that for every precovering graph $G$ in $W$ that is traversed by the infix $\rho_G$ of $\rho$, we have

$0 \leq \rho_G[first] \leq_\omega (q_r^G, cin)$ and $0 \leq \rho_G[last] \leq_\omega (q_f^G, cout)$.

## 3.2 Characteristic Equations

Goal: Capture $\mathbb{Z}$-intermediate acceptance with a system of inequalities.

Definition:

- We have variables $x[t]$ for the number of occurrences of transition $t$.
- We have variables $x[G, \substack{in \\ out}]$ for the counter valuation in the initial/final configuration of $G$.

$$
\begin{aligned}
Char(G) = \quad & x[t] \geq 0 \qquad \text{f.a. } t \in G.T \\
& \wedge\ Mark(G) \\
& \wedge\ Kirch(G) \\
& \wedge\ IAcc(G)
\end{aligned}
$$

as defined in the last lecture.

$$
\begin{aligned}
IAcc(G) = \quad & 0 \leq x[G, in] \leq_\omega G.cin \\
& \wedge\ 0 \leq x[G, out] \leq_\omega G.cout.
\end{aligned}
$$

// The constraint $\leq_\omega$ is simply true, if $G.cin[c] = \omega$ resp. $G.cout[c] = \omega$.

$$
\begin{aligned}
Char(G.up.W) = \quad & Char(G) \wedge Char(W) \\
& \wedge\ Mark(G, up, W)
\end{aligned}
$$

$$
Mark(G, up, W) = x[W[first], in] - x[G, out] = eff(up).
$$

We also need a homogeneous variant of the characteristic equation.

All we need to change is $IAcc(G)$ and $Mark(G, up, W)$.

Definition:

$$HomIAcc(G) = 0 \le x[G, in] \le O_{in} \leftarrow 0, \text{ where } G.c.in \text{ is concrete.}$$
$$\wedge\ 0 \le x[G, out] \le O_{out}$$

$$HomMark(G, up, W) = x[W[first], in] - x[G, out] = 0.$$