Recap:

↳ We have not yet given the definition of VASS.

A vector addition system with states (VASS) is a tuple

$$V = (Q, \Sigma, C, T)$$

with
- Q a finite set of states,
- Σ a finite alphabet (alphabets are always finite unless stated otherwise),
- C a finite set of counters, and
- $T \subseteq Q \times \Sigma \times \underbrace{\mathbb{Z}^C}_{eff(t)} \times Q$ a finite set of transitions.

The semantics is as expected. We write $\to_{\mathbb{Z}}$ if counters are allowed to go negative.

↳ We learned about two important systems of equalities.

For $x \in \mathbb{N}^T$, $y_1, y_2 \in \mathbb{Z}^S$, we define

$$\forall q \in Q: \quad \sum_{t=(-,q)} x(t) - \sum_{t=(q,-)} x(t) = 0. \qquad (KIRCH(V))$$

$$y_2 - y_1 - \sum_{t \in T} x(t) \cdot eff(t) = 0 \qquad (MARK(V))$$

$$\begin{aligned} y_1 &= m_1 \\ y_2 &= m_2 \end{aligned} \qquad (ACC(m_1, m_2))$$

↳ Kirchhoff's inequality appeared in this result:

Theorem (Euler / Kirchhoff):

Let $G = (V, T)$ be a directed graph that is strongly connected.

Let $x \in \mathbb{N}^T$ satisfy $(KIRCH) \wedge x \geq 1$.

Then for every node $v \in V$ there is a cycle c rooted in v so that $\Psi(c) = x$.

The theorem yields a sufficient condition for $\mathbb{Z}$-reachability in strongly connected VASS.
We call a VASS strongly connected, if the underlying graph (drop the edge annotations) is.

<u>Lemma:</u>

Let V be a strongly connected VASS.
If

$$\boxed{x \geq 1} \wedge KIRCH(V) \wedge MARK(V) \wedge ACC(m_1, m_2)$$

is feasible over $\mathbb{Z}$, then

$$(q, m_1) \to_{\mathbb{Z}}^{*} (q, m_2), \text{ for all } q \in Q.$$

<u>Proof:</u>

The Euler-Kirchhoff-Theorem guarantees the existence of a path/cycle.
The marking equation + acceptance make sure the resulting counter values are as expected. $\square$

There is also a kind of converse of the Euler-Kirchhoff-Theorem.

<u>Lemma:</u>

Let $G = (V, T)$ be a directed graph.
Let c be a cycle in G.
Then $\Psi(c)$ satisfies $KIRCH(G)$.

<u>Proof:</u> We proceed by induction on the number of primitive cycles that c goes through.

IH: If c is a primitive cycle,

then it clearly satisfies $KIRCH(G)$.

IS: Assume c goes through $n+1$ primitive cycles.

Then it has the shape

$$c = r_1 . c' . r_2 ,$$

where c' is another primitive cycle and $r_1 . r_2$ a cycle.

We already know $\Psi(c')$ satisfies $KIRCH(G)$.

Moreover, $r_1 . r_2$ only goes through n primitive cycles.

Hence, also $\Psi(r_1 . r_2)$ satisfies $KIRCH(G)$

by the induction hypothesis.

As $KIRCH(G)$ is stable under addition,

and as

$$\Psi(c) = \Psi(r_1 . r_2) + \Psi(c'),$$

also $\Psi(c)$ satisfies $KIRCH(G)$. □

With this converse,

we obtain a necessary condition

for $\mathbb{Z}$-reachability in VASS.

Lemma:

Let V be a VASS with $(q, m_1) \to_{\mathbb{Z}}^{*} (q, m_2)$.

Let c be the sequence of transitions on this path.

Then $\Psi(c)$ satisfies $\boxed{x \geq 0} \wedge KIRCH(G) \wedge MARK(G) \wedge ACC(m_1, m_2)$.

Proof:

- $x \geq 0$ is trivial
- $KIRCH(G)$ is the above lemma.
- $MARK(G, m_1, m_2)$ holds by the lemma we had earlier. □

Ziel: Charakterisierung von $\mathbb{Z}$-Reachability.

Ansatz: Solve the equation system from the necessary condition (the one with $x \geq 0$),

add the missing transitions with cycles that have zero-effect.

Definition:

- Let V be a strongly connected VASS with counter valuations / markings $m_1, m_2 \in \mathbb{Z}^C$.
  The characteristic equation (for reachability of m_2 from m_1 in V) is
  $$CHAR(V, m_1, m_2) := x \geq 0 \wedge KIRCH(V) \wedge MARK(V) \wedge ACC(m_1, m_2).$$
- The homogeneous variant of the characteristic equation is
  $$CHAR(V, 0, 0).$$
- A support solution to the characteristic equation is a solution (x, y_1, y_2) to the homogeneous variant.
- The support of the characteristic equations is the set
  $$supp(CHAR(V, m_1, m_2)) := \{x(t) \mid \exists \text{ support solution } (x, y_1, y_2) \text{ with } x(t) > 0\}$$
- A full support solution is a support solution that gives a positive value to all variables in the support.

Lemma:
There always is a full support solution.

Proof: If (x', y_1', y_2') and (x'', y_1'', y_2'') are support solutions, so $(x', y_1', y_2') + (x'', y_1'', y_2'')$.

So for each variable $x(t) \in supp(CHAR(V, m_1, m_2))$,
we find a support solution that justifies this membership,
and we add them up. $\square$

<u>Proposition</u>:

Let $V = (Q, \Sigma, C, T)$ be a strongly connected VASS.

Let $m_1, m_2 \in \mathbb{Z}^S$ be counter valuations.

Assume all transitions are in the support,

meaning

$$supp(CHAR(V, m_1, m_2)) = \{x(t) \mid t \in T\}.$$

Then $\quad (q, m_1) \to^* (q, m_2) \quad$ for all $q \in Q$

iff $\quad CHAR(V, m_1, m_2)$ is feasible over $\mathbb{Z}$.

<u>Proof</u>:

"$\Rightarrow$" This is the necessary condition from above.

"$\Leftarrow$" Let (x, y_1, y_2) be the solution
that we assume would exist.
Let $(\bar{x}, \bar{y}_1, \bar{y}_2)$ be the full support solution
that exists by the above lemma.
By definition, $\bar{y}_1 = \bar{y}_2 = 0$.
Basic linear algebra shows that

$$Ax = b \wedge Ay = 0 \Rightarrow A(x+y) = b,$$

since matrix multiplication distributes over addition.
Hence, $(x, y_1, y_2) + (\bar{x}, \bar{y}_1, \bar{y}_2)$ solves $CHAR(V, m_1, m_2)$.

Since $(\bar{x}, \bar{y}_1, \bar{y}_2)$ is a full support solution,
we have $(x + \bar{x})(t) > 0$ for all $t \in T$.
The lemma on the sufficient condition now applies,
and yields the desired conclusion. $\square$

<u>Remark</u>:

This is <u>not</u> how you actually solve $\mathbb{Z}$-reachability in VASS.
Instead, given (q_1, m_1) and (q_2, m_2),
one computes the Parikh image from q_1 to q_2, $\Psi(\mathcal{L}(q_1, q_2))$.
The Parikh image will be a set of vectors from $\mathbb{N}^T$.
Then we check whether there is a Parikh vector
whose effect is $m_2 - m_1$:

$$\exists x. \quad x \in \Psi(\mathcal{L}(q_1, q_2)) \wedge \sum x(t) \cdot \mathit{eff}(t) = m_2 - m_1.$$

The point is that one can compute
a formula in existential Presburger arithmetic
for $\Psi(\mathcal{L}(q_1, q_2))$,
and therefore the whole query is in <u>existential Presburger</u>.
Since the formula can be computed in linear time, $\mathbb{Z}$-reachability is in NP.
The problem is NP-hard even if numbers are encoded in unary
[Haase, Halfon '14].

Why the above approach?
Because it fits to the solution to $(\mathbb{N}$-$)$ reachability.
It will help to understand the support semantically.

Idea: What we have seen in the above argument is this:

- If a variable is in the support, then we can add a support solution to a given solution to increase its value.
- We can do this repeatedly, and therefore variables in the support can have arbitrarily high values in the solution space.

The reverse is also true:

- If a variable is unbounded in the solution space, then it has to be in the support.

Lemma:
Consider $Ax = b$ feasible over $\mathbb{N}$ and a variable $x(t)$.
There are $\mathbb{N}$-solutions x with arbitrarily high values for $x(t)$
iff there is an $\mathbb{N}$-solution y to $Ay = 0$ with $y(t) > 0$.

Proof:
„$\Leftarrow$" by the above argument.
„$\Rightarrow$" Since there are solutions with arbitrarily high values for $x(t)$, we can come up with an infinite sequence

$x_1, x_2, x_3 \dots$

in which $x(t)$ is unbounded.
We then form the subsequence $(x_{e(i)})_{i \in \mathbb{N}}$ as follows:

$e(0) = 0$, $e(i+1)$ = the first moment $x(t)$ has a value strictly higher than $x_{e(i)}(t)$.