Recall:

Marking Equation:

$M_0 [\sigma\rangle M_1$ implies $M_1 = M_0 + \mathbb{C}\,\Psi(\sigma)$.

Consequence:

- If $M_1 - M_0 = \mathbb{C}x$ does not have a solution in $\mathbb{N}^T$, then $M_1 \notin Reach(M_0)$.

- If $M = \mathbb{C}x$ does have a solution in $\mathbb{N}^T$, then there is a marking $M_1 \in \mathbb{N}^S$ with $M_1 + M \in Reach(M_1)$.

  // If you make M_1 large enough, you can for sure execute the Parikh vector.

Corollary:

If $\mathbb{C}x \gneq 0$ has a solution in $\mathbb{N}^T$, then there is a marking M_1 from which the Petri net is unbounded.*

Structural Invariants:

$I \in \mathbb{Z}^S$ with $C^T \cdot I = 0$.

Lemma: $M_1 \in Reach(M_0)$ implies $I^T \cdot M_1 = I^T \cdot M_0$.

Corollary: If $I \geq 1$ is an S-invariant, then the PN is bounded* from every initial marking.

*Definition: A PN is bounded from M_0, if $\exists b \in \mathbb{N}.\ \forall M \in Reach(M_0).\ \forall s \in S.\ M(s) \leq b$.

1.2 Transition Invariants

Definition:

A transition invariant of $N = (S, T, W)$ is a vector $J \in \mathbb{N}^T$ with $\mathbb{C} \cdot J = 0$.

Lemma:

Let $M_0 [\sigma\rangle M_1$ in N.

Then $M_0 = M_1$ iff $\Psi(\sigma)$ is a transition invariant.

Corollary:

Petri nets without transition invariants have acyclic reachability graphs.

1.3 Euler-Kirchhoff-Equations

Goal:
- The marking equation is a necessary condition for the existence of a path.
- Our goal is to develop a sufficient one, for now without counters.

Lemma (Euler-Kirchhoff):

Let $G = (V, E)$ be a strongly connected directed graph.

Let $x : E \to \mathbb{N}$ with
- $x \geq 1$
- $\underbrace{\forall v \in V.\ \sum_{e=(w,v)} x(e) = \sum_{e=(v,w)} x(e)}_{\text{Kirchhoff equations.}}$

Then for every node $v \in V$ there is a cycle c rooted in v with $\Psi(c) = x$.

Proof:

We proceed by induction on the number of primitive cycles in G.

$\# = 1$: ✓

$\# > 1$: Pick a primitive cycle c' rooted in v.

Form $x' = x - \Psi(c')$.

- If $x'(e) = 0$, remove edge e from G. Denote the result by $G[x']$.

Claim:

(1) $\forall v \in V. \sum_{e=(u,v)} x'(e) = \sum_{e=(v,w)} x'(e)$.

(2) $G[x']$ is a set of SCCs.

Ad (1):

x and $\Psi(c')$ satisfy Kirchhoff, and so does $x - \Psi(c')$.

Ad (2):

$\xrightarrow[0]{///} \cdot \xrightarrow{\neq 0}$ would violate Kirchhoff in (1).

- If we do not remove an edge, we repeat the procedure.
- If we remove edges, $G[x']$ has less primitive cycles. We decompose $G[x']$ into its SCCs, say G_1 and G_2. Let v_i be a node in G_i that is connected to an edge which has been removed.

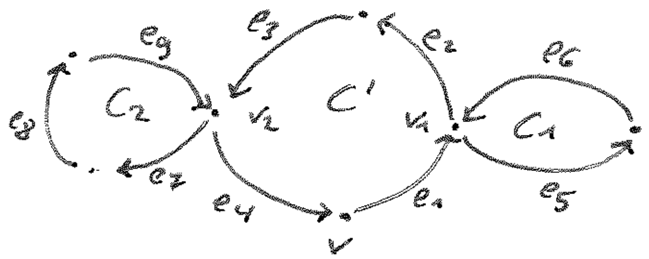
- We invoke the IH on G_i and v_i, which yields a cycle c_i rooted in v_i.
- We define

$$c = c'[{}^{c_1, c_2}/_{v_1, v_2}]$$ // Glue cycles into an occurrence of v_i

as the cycle rooted in v. □

<u>Illustration:</u>

$G =$



$c' = e_1 e_2 e_3 e_4$

$c_1 = e_5 e_6$

$c_2 = e_7 e_8 e_9$

$c = c'[{}^{c_1, c_2}/_{v_1, v_2}]$

$= e_1 . c_1 . e_2 . e_3 . c_2 . e_4 .$

<u>Perspective:</u>

There is more structure theory of Petri nets.

<u>Definition:</u>

- A <u>trap</u> is a set of places $Q \subseteq S$ with $Q^\bullet \subseteq {}^\bullet Q$

  $Q^\bullet = \{t \in T \mid \exists s \in Q.\ W(s,t) > 0\}$

  // If a transition removes a token from Q, it also inserts a token into Q.
- A <u>siphon</u> is a set of places $Q \subseteq S$ with ${}^\bullet Q \subseteq Q^\bullet$.

<u>Lemma:</u>

Consider (S, T, W) with Q_1 a trap and Q_2 a siphon.

Let $M_0 \in \mathbb{N}^S$ with $M_0(Q_1) > 0$
and $M_0(Q_2) = 0$.

Let $M_1 \in Reach(M_0)$.

Then $M_1(Q_1) > 0$ and $M_1(Q_2) = 0$.

Based on the marking equation, S-invariants, traps, and siphons, one can build linear-algebraic (constraint-based verification techniques for Petri nets.
(not falsification)