Where are we?

- NBA-languages
  = ω-regular languages
- closed under $\cap, \cup, {}^{-}$

DBA-languages $\neq \emptyset$ — finitely many bs
- not closed under $^{-}$

Decision Procedures:

Goal of automata constructions:
- ↳ solve model checking $A \models \varphi$
- ↳ solve validity / satisfiability of an MSO formula φ.

Here: consider corresponding algorithmic problems
- ↳ Emptiness: $L(A) = \emptyset$?
- ↳ Universality: $L(A) = \Sigma^{\omega}$?
- ↳ Inclusion: $L(A) \subseteq L(B)$?

Even if the problems are mutually encodable, dedicated decision procedures make sense:
complementation is expensive

Inclusion - the Standard Way

Lemma:
For a given NBA A, it is decidable in polynomial time whether $L(A) = \emptyset$.

- Reduce $L(A) \subseteq L(B)$ to emptiness
- Remember: ↳ NBA-languages are closed under complementation
  ↳ Therefore, all operations used in the following equivalences can be computed.

$$L(A) \subseteq L(B) \quad \text{iff}_{\text{(set theory)}} \quad L(A) \cap \overline{L(B)} = \emptyset \quad \text{iff}_{\text{(construction } \overline{B})} \quad L(A) \cap L(\overline{B}) = \emptyset$$
$$\text{iff}_{\text{(construction } \times)} \quad L(A \times \overline{B}) = \emptyset.$$

Lemma:

Inclusion $L(A) \subseteq L(B)$ holds iff $L(A \times B) = \emptyset$.

Therefore, the inclusion problem is decidable.

Universality with Ramsey:

Consequence of Büchi's complementation procedure

Theorem (Fogarty & Vardi '10)

Consider an NBA A.

We have

$L(A) = \Sigma^\omega$ iff

for all $[u]_{\sim A}, [v]_{\sim A}$ with $[v]_{\sim A} \cap \Sigma^+ \neq \emptyset$

and $[uv]_{\sim A} = [u]_{\sim A}$ and $[vv]_{\sim A} = [v]_{\sim A}$

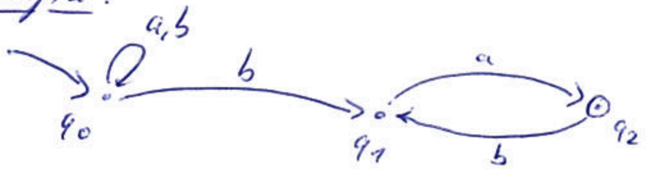
there is $q \in Q$ with $(q_0, q) \in R_{[u]_{\sim A}}$ and $(q, q) \in R^{fin}_{[v]_{\sim A}}$

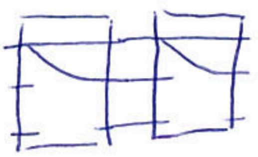
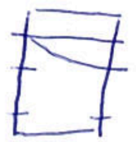
Algorithmically:

- Use theorem in contraposition to disprove universality.
- Go through idempotent equivalence classes $[vv]_{\sim A} = [v]_{\sim A}$
  - ↳ Is there $[u]_{\sim A}$ with $[uv]_{\sim A} = [u]_{\sim A}$ so that for all $q \in Q$:

    $q_0 \xrightarrow{u} q$ implies $q \not\xrightarrow{v}_{fin} q$?
- Then A does not accept all words.

Example:



- We have Box(bb.bb) =  =  = Box(bb).
- Thus, $[bb.bb]_{\sim A} = [bb]_{\sim A}$.
- Select $[u]_{\sim A} = [v]_{\sim A} = [bb]_{\sim A}$

This means

$[uv]_{\sim A} = [u]_{\sim A}$ and $[vv]_{\sim A} = [v]_{\sim A}$.

We have

- $q_0 \xrightarrow{bb} q_0$ but $q_0 \not\xrightarrow{v}_{fin} q_0$
- $q_0 \xrightarrow{bb} q_1$ but $q_1 \not\xrightarrow{v}_{fin} q_1$.

Indeed, $bb.(bb)^\omega \notin L(A)$ and thus $L(A) \neq \Sigma^\omega$.

<u>Remark:</u>

↳ We can stop the construction of equivalence classes when the first counterexample to universality has been found

(How to find counterexamples systematically?)

↳ If $L(A) = \Sigma^\omega$, have to construct full multiplication table

(How to find $[uv]_{\sim A} = [u]_{\sim A}$?)

(How to stop the algorithm earlier in case of success?)

↳ The algorithm seems to work well in parallel.

⇒ Bachelor's and Master's theses.

<u>Proof (of Fogarty & Vardi)</u>

„⇒" Let $A = (\Sigma, Q, q_0, \rightarrow, Q_F)$ with $L(A) = \Sigma^\omega$.

Consider classes $[u]_{\sim A}$ and $[v]_{\sim A}$ with $[v]_{\sim A} \cap \Sigma^+ \neq \emptyset$

and $[uv]_{\sim A} = [u]_{\sim A}$ and $[vv]_{\sim A} = [v]_{\sim A}$.

We have to find $q \in Q$ so that

$$(q_0, q) \in R_{[u]_{\sim A}} \quad \text{and} \quad (q,q) \in R^{fin}_{[v]_{\sim A}}.$$

Assume wlog. that $v \neq \varepsilon$.

By universality of A, A has an accepting run on $u.v^\omega$.

By the pigeonhole principle, some state $q \in Q$ is visited infinitely often along this run:

$$q_0 \xrightarrow{u.v^{i_0}} q \xrightarrow{v^{i_1}} q \xrightarrow{v^{i_2}} q \dots \quad \text{for } i_0, i_1, i_2, \dots > 0.$$

Because the run is accepting, there are infinitely many final states in between.

So we can assume

$$q_0 \xrightarrow{u.v^{i_0}} q \xrightarrow[fin]{v^{i_1}} q \xrightarrow[fin]{v^{i_2}} q \dots \quad \text{for } i_0, i_1, i_2, \dots > 0.$$

Since $[uv]_{\sim A} = [u]_{\sim A}$, we have $[uv^{i_0}]_{\sim A} = [u]_{\sim A}$.

↳ Thus, $(q_0, q) \in R_{[u]_{\sim A}}$.

Similarly, by $[vv]_{\sim A} = [v]_{\sim A}$ we have $[v^{i_j}]_{\sim A} = [v]_{\sim A}$ for all $j > 0$.

↳ Thus, $(q, q) \in R^{fin}_{[v]_{\sim A}}$.

"$\Leftarrow$" Assume all classes $[u]_{\sim A}, [v]_{\sim A}$ with $[v]_{\sim A} \cap \Sigma^+ \neq \emptyset$ and $[uv]_{\sim A} = [u]_{\sim A}$ and $[vv]_{\sim A} = [v]_{\sim A}$ have a state q as required.

↳ We have to prove universality, $L(A) = \Sigma^\omega$ (actually $L(A) \supseteq \Sigma^\omega$)

↳ Let $w \in \Sigma^\omega$. We show that $w \in L(A)$.

To this end, have to construct an accepting run.

Let $w = a_0 a_1 a_2 \dots$

<u>Apply Ramsey's theorem</u>:

Color $(\mathbb{N}, E)$ with the $\sim A$ equivalence classes:

$$f(\{i,j\}) := [a_i \dots a_{j-1}]_{\sim A} \quad \text{for } i < j.$$

With Ramsey's theorem, there is an infinite subset $S \subseteq \mathbb{N}$ and a class $[v]_{\sim A}$ so that

$$f(\{i,j\}) = [v]_{\sim A} \quad \text{for all } i < j \text{ in } S.$$

Then

$$\in S$$

$$w = a_0 \dots a_{i_0-1} \underbrace{\underbrace{a_{i_0} \dots a_{i_1-1}}_{\sim_{\mathcal{A}} v} \; \underbrace{a_{i_1} \dots a_{i_2-1}}_{\sim_{\mathcal{A}} v}}_{\sim_{\mathcal{A}} v} a_{i_2} \dots ,$$

i.e., the indices i_j belong to the set S.

Select $u := a_0 \dots a_{i_1-1}$.

$$\begin{aligned}
\text{Then } [u.v]_{\sim_{\mathcal{A}}} &= [a_0 \dots a_{i_0} \dots a_{i_1-1} . v]_{\sim_{\mathcal{A}}} \\
&= [a_0 \dots \underbrace{a_{i_0} \dots a_{i_1-1} . \overbrace{a_{i_1} \dots a_{i_2-1}}}]_{\sim_{\mathcal{A}}} \\
&= [a_0 \dots a_{i_0-1} . v]_{\sim_{\mathcal{A}}} \\
&= [a_0 \dots a_{i_0-1} \overbrace{a_{i_0} \dots a_{i_1-1}}]_{\sim_{\mathcal{A}}} = [u]_{\sim_{\mathcal{A}}} .
\end{aligned}$$

For $[vv]_{\sim_{\mathcal{A}}} = [v]_{\sim_{\mathcal{A}}}$, the argumentation is similar.
By the assumption on the equivalence classes, there is $q \in Q$ with

$$(q_0, q) \in R_{[u]_{\sim_{\mathcal{A}}}} \quad \text{and} \quad (q,q) \in R^{fin}_{[v]_{\sim_{\mathcal{A}}}}$$

This yields an accepting run of $\mathcal{A}$ on w. $\square$